Second round
Dutch Mathematical Olympiad

Friday 10 March 2023

Solutions

B-problems

B1. 231 We can factor both the numerator and denominator of the fraction into prime factors. Factors appearing in both the numerator and denominator cancel each other out. What remains is the simplified form for the fraction. The prime factors that can occur are 2, 3, 5, 7 and 11. For each of these prime numbers, we keep track of how many times they occur as a prime factor in the numbers 1 up to and including 12. If a prime number occurs an odd number of times, the factors cannot all cancel each other out and the prime factor must occur in the smallest possible integer we are looking for.

The prime numbers 7 and 11 occur only once, namely in the numbers 7 and 11 themselves. The prime number 5 occurs twice: once as 5 and once as a factor of 10. The prime number 3 occurs five times: once as a factor of each of the numbers 3, 6 and 12 and twice as a factor of the number 9. The prime number 2 occurs 10 times: once as a factor in the numbers 2, 6 and 10, twice as a factor in the numbers 4 and 12 and three times as a factor in the number 8.

We see that the integer we are looking for must have at least factors 3, 7 and 11 and thus must be at least $3 \times 7 \times 11 = 231$. It turns out this is in fact possible: we can get 231 by taking

$$\frac{4 \times 7 \times 9 \times 10 \times 11 \times 12}{1 \times 2 \times 3 \times 5 \times 6 \times 8}.$$

B2. $6\sqrt{2} - 6$ We first consider triangle BRQ. That triangle is isosceles, because QR is parallel to the diagonal AC of the square. Moreover, $|QR| = 6$. So, using the Pythogorean theorem, we find that $2|BQ|^2 = |BQ|^2 + |BR|^2 = |QR|^2 = 36$. We thus find that $|BQ|^2 = 18$ and $|BQ| = 3\sqrt{2}$. Similarly, we find that $|DP| = 3\sqrt{2}$. It then follows that $|AQ| = 6 - |BQ| = 6 - 3\sqrt{2}$ and similarly $|AP| = 6 - 3\sqrt{2}$. The Pythagorean theorem now gives that $|PQ|^2 = |AP|^2 + |AQ|^2 = 2(6 - 3\sqrt{2})^2$ and so $|PQ| = \sqrt{2} \cdot (6 - 3\sqrt{2}) = 6\sqrt{2} - 6$. Since $ABFE$ and $PSRQ$ have equal dimensions, this is also the length of AE.

B3. 20406080 Clearly, n must be at least 12345678. To find the smallest possible n, we first try with eight-digit numbers. Write n as $\overline{\texttt{abcdefgh}}$, the number with the digits $\texttt{a}, \ldots, \texttt{h}$. The crab of n is then $k = \overline{\texttt{hgfedcba}}$ and $n - k = 12345678$.

$$\begin{array}{l} \texttt{abcdefgh} \\ \underline{\texttt{hgfedcba}} - \\ \texttt{12345678} \end{array}$$

We get the smallest possible n by taking the first digit as small as possible. We try $\texttt{a} = 1$. If we then look at the first digit of $n - k$, we see that $\texttt{h} = 0$ must hold, but then the last digit of $n - k$ equals 9 and that is not correct. So we see that $\texttt{a} \geqslant 2$ must hold.

We take $\texttt{a} = 2$ and for the second digit we try the smallest possible digit: $\texttt{b} = 0$. By looking at the last two digits of $n - k$, we see that $\texttt{g} = 8$ and $\texttt{h} = 0$ must hold. So far, we thus have

$$\begin{array}{l} \texttt{20cdef80} \\ \underline{\texttt{08fedc02}} - \\ \texttt{12345678} \end{array}$$

Because the first two and last two digits in the subtraction are already correct, without needing to 'borrow' a 1 from the second digit, we will now attempt to find $\mathtt{c}, \mathtt{d}, \mathtt{e}$, and $\mathtt{f}$ such that $\overline{\mathtt{cdef}} - \overline{\mathtt{fedc}} = 3456$.

The smallest possible $\mathtt{c}$ we can now choose is $\mathtt{c} = 3$. If we do that, we find $\mathtt{f} = 0$ by looking at the third digit of $n - k$. The sixth digit of $n - k$ then becomes a 7, which is not correct. So we see that $\mathtt{c} \geqslant 4$ must hold.

We take $\mathtt{c} = 4$ and the smallest possible $\mathtt{d}$, namely $\mathtt{d} = 0$. By looking at the fifth and sixth digits of $n - k$, we find that $\mathtt{e} = 6$ and $\mathtt{f} = 0$ must hold. We now find $n = 20406080$ which indeed is a solution, since $20406080 - 8060402 = 12345678$.

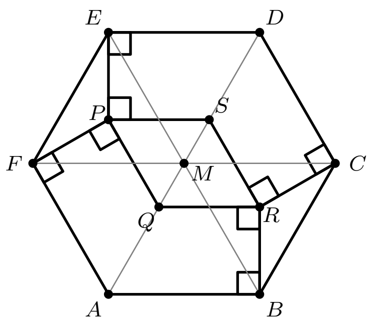
B4. 9 Let a, b and c be the digits of n; that is, $n = 100a + 10b + c$. Then the number of participant numbers from 1 to n ending in a 5 equals $10a + b$ if $c < 5$ and equals $10a + b + 1$ if $c \geqslant 5$. The number formed by the last two digits of n is $10b + c$.

First, consider the case $c < 5$. Then $10a + b$ must be equal to $10b + c$. These are both two-digit numbers, one with digits a and b, the other with digits b and c. We find that $a = b = c \neq 0$ and we get the solutions $n = 111, 222, 333, 444$.

The case $c \geqslant 5$ is slightly more complicated. The number $10a + b + 1$ must be equal to $10b + c$. If $b < 9$, then the digits of $10a + b + 1$ are exactly a and $b + 1$ and we find $a = b$ and $b + 1 = c$. This gives the solutions $n = 445, 556, 667, 778, 889$. If $b = 9$, then we get $10a + 10 = 90 + c$. It follows that $c = 0$ must hold, which contradicts $c \geqslant 5$.

In total, we have found 9 possible values for n.

B5. $\frac{8}{9}$ First of all, note that the quadrilateral $PQRS$ is a parallelogram, because its opposite sides are pairwise parallel. For example, PS is parallel to ED, because of the right co-interior angles at E and P. Similar reasoning gives that SR is parallel to DC, QR is parallel to AB (and therefore to ED), and PQ is parallel to FA (and therefore to DC).



In the picture on the right, the three diagonals of the hexagon are drawn as auxiliary lines. These divide the hexagon into six equilateral triangles of area 1. We call the centre of the hexagon M. Since AD is an axis of symmetry of the figure, the points Q and S lie on the diagonal AD. Because of parallelism, the diagonals BE and CF divide the parallelogram $PQRS$ into four small parallelograms each having $\frac{1}{4}$ of the area of $PQRS$. We also see that such a small parallelogram has twice the area of a small equilateral triangle.

We can find the size of such a small triangle as follows. Triangle BCM consists of three smaller triangles with equal dimensions: BCR, CMR and BMR. Each of these small isosceles triangles has an area of $\frac{1}{3}$. Therefore, the altitude of the vertex R considered from the base CM is exactly $\frac{1}{3}$ of the total altitude of the triangle. Because of parallelism, vertex Q also lies on $\frac{1}{3}$ of the side MA seen from M. Thus, the small triangle with side QM has area $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ of the area of the big equilateral triangle, which means that is has area $\frac{1}{9}$. It follows that the parallelogram $PQRS$ has area $8 \times \frac{1}{9} = \frac{8}{9}$.

C-problems

C1. *We will first look at the problem for a smaller number of blocks with consecutive numbers. With 1 block, only 1 tower is possible. With 2 blocks, both towers (top) 1–2 (bottom) and 2–1 are possible. With 3 blocks, these are the possible towers: 1–2–3, 2–1–3, 1–3–2, and 3–2–1. With 4 blocks, these are the possible towers:*

$$1-2-3-4, \quad 1-2-4-3, \quad 2-1-3-4, \quad 2-1-4-3,$$
$$1-3-2-4, \quad 1-4-3-2, \quad 3-2-1-4, \quad 4-3-2-1.$$

It seems that the number of towers doubles each time. We are going to prove this. Starting from a tower with $n > 1$ blocks, consider the block with the highest number, n. If this block lies on top of another block, then this must be a block with a lower number (as there does not exist a block with a higher number), and according to the requirements it has to lie on the block with number $n-1$. Hence, block n lies either completely on the bottom, or directly on top of block $n-1$. Because all other blocks can also lie on top of block $n-1$, we see that after removing block n, we are left with a valid tower of $n-1$ blocks.

Now we want to show that from any (valid) tower of $n-1$ blocks we can make a valid tower of n blocks. In every (valid) tower of $n-1$ blocks we can insert block n either at the bottom or directly on top of block $n-1$; other places are not allowed and in this way block n never lies on a block with a number that is too small. In both cases the tower of n blocks is a valid one, because on top of block n any other block is allowed. So from any valid tower of $n-1$ blocks we can make two valid towers of n blocks.

So we see that the number of towers indeed doubles every time. For $n = 1$ there is $1 = 2^0$ tower, for $n = 2$ there are $2 = 2^1$ towers and in general there are 2^{n-1} towers. So the number of towers with 10 blocks is $2^9 = 512$.

C2. (a) Suppose the two positive integers are $n-10$ and $n+10$, and hence $n > 10$. Then the product is equal to $(n-10)(n+10) = n^2 - 100$ and we are looking for an $n > 10$ such that $n^2 - 100 + 23$ ends in the digits 23. But that means we want n^2 to end in the digits 00. In other words, we want n^2 to be divisible by 100 and thus n to be divisible by 10. The smallest possible solution is $n = 20$ and we see that $10 \cdot 30 + 23 = 323$ does indeed end at 23. So the smallest possible outcome is 323.

(b) We take again the integers $n-10$ and $n+10$. Now we need to find, for a certain integer k, a solution for $n^2 - 100 + 23 = k^2$, or $n^2 = k^2 + 77$. The difference between two consecutive squares is an odd number that becomes 2 bigger every time. We have that $1^2 - 0^2 = 1$, $2^2 - 1^2 = 3$, $3^2 - 2^2 = 5$, et cetera. In general: $(m+1)^2 - m^2 = 2m + 1$. We can get 77 by taking $m = \frac{76}{2} = 38$ and so $k = 38$ and $n = 39$. We see that indeed it holds that $29 \cdot 49 + 23 = 1444 = 38^2$. So it turns out to be possible that the result is a square. *In fact, it turns out that this solution is unique, but the problem did not ask us to prove that.*