Final round
Dutch Mathematical Olympiad

Friday 16 September 2022
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. A positive integer \( n \) is called divisor primary if for every positive divisor \( d \) of \( n \) at least one of the numbers \( d - 1 \) and \( d + 1 \) is prime. For example, 8 is divisor primary, because its positive divisors 1, 2, 4, and 8 each differ by 1 from a prime number (2, 3, 5, and 7, respectively), while 9 is not divisor primary, because the divisor 9 does not differ by 1 from a prime number (both 8 and 10 are composite).

(a) Which odd numbers can occur as the divisor of a divisor primary number?
(b) Determine the largest divisor primary number.

2. A set consisting of at least two distinct positive integers is called centenary if its greatest element is 100. We will consider the average of all numbers in a centenary set, which we will call the average of the set. For example, the average of the centenary set \{1, 2, 20, 100\} is \( \frac{123}{4} \) and the average of the centenary set \{74, 90, 100\} is 88.
Determine all integers that can occur as the average of a centenary set.

3. Given a positive integer \( c \), we construct a sequence of fractions \( a_1, a_2, a_3, \ldots \) as follows:

- \( a_1 = \frac{c}{c+1} \);
- to get \( a_n \), we take \( a_{n-1} \) (in its most simplified form, with both the numerator and denominator chosen to be positive) and we add 2 to the numerator and 3 to the denominator. Then we simplify the result again as much as possible, with positive numerator and denominator.

For example, if we take \( c = 20 \), then \( a_1 = \frac{20}{21} \) and \( a_2 = \frac{22}{23} = \frac{11}{12} \). Then we find that \( a_3 = \frac{13}{15} \) (which is already simplified) and \( a_4 = \frac{15}{18} = \frac{5}{6} \).

(a) Let \( c = 10 \), hence \( a_1 = \frac{10}{11} \). Determine the largest \( n \) for which a simplification is needed in the construction of \( a_n \).
(b) Let \( c = 99 \), hence \( a_1 = \frac{99}{100} \). Determine whether a simplification is needed somewhere in the sequence.
(c) Find a value of \( c \) for which in the first step of the construction of \( a_5 \) (before simplification) the numerator and denominator are divisible by 5.
4. In triangle $ABC$, the point $D$ lies on segment $AB$ such that $CD$ is the angle bisector of angle $C$. The perpendicular bisector of segment $CD$ intersects the line $AB$ in $E$. Suppose that $|BE| = 4$ and $|AB| = 5$.

(a) Prove that $\angle BAC = \angle BCE$.
(b) Prove that $\frac{|AE|}{|CE|} = \frac{|CE|}{|BE|}$.
(c) Prove that $2|AD| = |ED|$.

5. Kira has 3 blocks with the letter A, 3 blocks with the letter B, and 3 blocks with the letter C. She puts these 9 blocks in a sequence. She wants to have as many distinct distances between blocks with the same letter as possible. For example, in the sequence ABCAABCBC the blocks with the letter A have distances 1, 3, and 4 between one another, the blocks with the letter B have distances 2, 4, and 6 between one another, and the blocks with the letter C have distances 2, 4, and 6 between one another. Altogether, we got distances of 1, 2, 3, 4, and 6; these are 5 distinct distances.

What is the maximum number of distinct distances that can occur?