Final round
Dutch Mathematical Olympiad

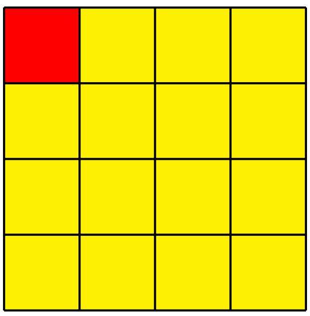
Friday 17 September 2021

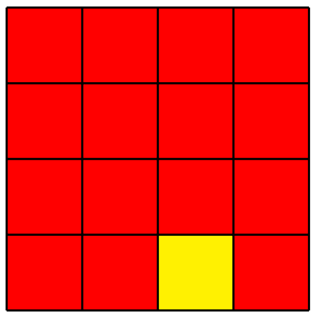
Solutions

1. (a) First note that there are $3 \cdot 4 = 12$ horizontal borders between two cards, and also 12 vertical borders. Suppose that k of these borders count as -1, then there are $24 - k$ borders counting as $+1$. This gives a monochromaticity of $(24 - k) \cdot (+1) + k \cdot (-1) = 24 - 2k$. Hence, the monochromaticity is always an even number.

If all cards have the same colour, then all borders count as $+1$, and we get the maximal monochromaticity of 24. Can 22 also occur as the monochromaticity? No, and we will prove that by contradiction. Suppose there is an assignment of cards having monochromaticity 22. Then there has to be one border with -1 and the rest must count as $+1$. In other words, all adjacent cards have the same colour, except for one border. Consider the two cards at this border, and choose two adjacent cards so that you obtain a 2×2 square. For each pair of cards, you can find such a 2×2 square. If you start on the left top and go around the four cards in a circle (left top – right top – right bottom – left bottom – left top), then you cross four borders. Since you are starting and ending in the same colour, you must have crossed an even number of borders where the colour is changing. This, however, is in contradiction with the assumption that there is only one border at which the two cards have different colours. We conclude that the monochromaticity can never be 22.

The next possibilities for large monochromaticites are 20 and 18. Then there have to be 2 or 3 borders between cards of different colours. This can be achieved by the following colourings:

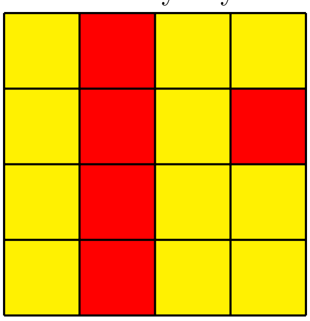


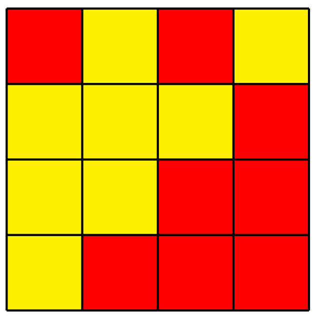


Altogether, the three largest numbers on Niek's list are 24, 20, and 18. □

(b) Suppose that we put the cards such that the monochromaticity is x. Then we can turn half of the cards, as in a chess board pattern: we turn a card if and only if all of the adjacent cards are not turned. With this operation all borders between cards change sign, and we obtain a monochromaticity of $-x$. In other words, x is a possible value for the monochromaticity if and only if $-x$ is possible. Therefore, the three smallest numbers on Niek's list are the negatives of the three greatest numbers: -24, -20, and -18. □

(c) We already proved that the monochromaticity is always an even number. The smallest possible positive even number is 2. This monochromaticity can be obtained by having 13 borders between squares of the same colour, and 11 borders between squares of different colours. There are many ways to achieve this, for example:





□

2. Version for klas 5 & klas 4 and below

(a) Suppose towards a contradiction that we can find three teams in a balanced tournament that all play against each other, say teams A, B and C. Because $n \geqslant 5$ there are two other teams, say D and E. Since A, B and C already play three matches between them, there are no other matches between the quadruple A, B, C and D. In other words: D does not play against A, B and C. The same holds for team E. If we now consider the quadruple A, B, D and E we see that there are at most two matches: A against B, and possibly D against E. This means that we have found four teams such that there are not exactly three matches between these four teams. This is a contradiction. □

(b) We will first show that a balanced tournament is not possible with $n \geqslant 6$ teams. Then we give an example of a balanced tournament for $n = 5$. This shows that 5 is the largest value of n for which a balanced tournament with n teams exists.

Suppose that $n \geqslant 6$ and, towards a contradiction, that a balanced tournament with n teams exists. We look at the first six teams, say teams A to F. Suppose that A plays against at most two of these teams, say at most against B and C but not against D, E and F. Since three matches have to be played among the quadruple A, D, E and F, the teams D, E and F all have to play against one another. This is a contradiction with part (a).

We conclude that A has to play against at least three of the teams, for example B, C and D. This gives three matches in the quadruple A, B, C, D, so B, C and D do not play any matches between them. Because the quadruple B, C, D, E also has to play three matches, E has to play against all of B, C and D. But now we find a contradiction in the quadruple A, B, C, E: there are already four matches between these teams (A against B, A against C, B against E, and C against E). Therefore a balanced tournament with $n \geqslant 6$ does not exist.

To make a tournament with five teams, imagine the teams are standing in a circle. Two teams play against each other if they are standing next to each other in the circle. If we look at any quadruple of teams, we see there are exactly three pairs of teams standing next to each other in the circle. So the four teams plays three matches between them. We conclude that 5 is the largest value of n for which a balanced tournament with n teams exists. □

2. Version for klas 6

In the solution for klas 5 & klas 4 and below this problem is solved in two steps. In part (a) it is proven that in a balanced tournament with $n \geqslant 5$ teams, there are no three teams that all play against one another in the tournament. This is then used in part (b) to prove that 5 is the largest value of n for which a balanced tournament with n teams exists. □

3. Version for klas 5 & klas 4 and below

(a) When the frog finished n jumps, it made $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ steps in total. To get back at 0, the frog must make the same number of steps to the left and the right. Thus, the total number of steps must be even. This means that $\frac{1}{2}n(n+1)$ is even, and hence $n(n+1)$ is a multiple of four. This yields that n or $n+1$ must be a multiple of four, that is, n is of the shape $n = 4k - 1$ or $n = 4k$. Potential values for n are $3, 4, 7, 8, 11, 12, \ldots$. Now we will show that for each of these values of n the frog can get back at 0 after n jumps.

We will prove this by induction. For $n = 3$ and $n = 4$, it is not hard to find a solution: $1 + 2 - 3 = 0$ and $1 - 2 - 3 + 4 = 0$. Now suppose that we can choose pluses and minuses such that $\pm 1 \pm 2 \pm \cdots \pm m = 0$ for a certain integer m. Then we can also find a combination of pluses and minuses such that $\pm 1 \pm 2 \pm \cdots \pm (m+4) = 0$. Indeed:

$$\begin{aligned} &\pm 1 \pm 2 \pm \cdots \pm m + (m+1) - (m+2) - (m+3) + (m+4) \\ &= 0 + (m+1) - (m+2) - (m+3) + (m+4) \\ &= 1 - 2 - 3 + 4 \\ &= 0. \end{aligned}$$

It follows that the frog can indeed get back to 0 after n jumps for each n of the shape $n = 4k - 1$ or $n = 4k$. □

(b) This problem actually consists of two variants on part (a), namely in the horizontal and the vertical direction. We start by considering the vertical direction. The frog is making jumps consisting of even numbers of steps. This is actually what was happening in part (a), except that the jumps are twice as long. Hence, the frog can end up on the x-axis if the last jump in the vertical direction consists of $8k - 2$ or $8k$ steps. Now we have to investigate whether the frog can also arrive back on the y-axis, and hence at the origin $(0, 0)$. The last horizontal jump is one before or one after the last vertical jump, hence the last horizontal jump must consist of $8k - 3$, $8k - 1$, or $8k + 1$ steps.

We will investigate whether it is possible that $\pm 1 \pm 3 \pm \cdots \pm n = 0$ for n of the shape $8k - 3$, $8k - 1$, or $8k + 1$. To get back to the y-axis, the total number of horizontal steps must be even. Because each jump consists of an odd number of steps, the frog must make an even number of jumps in the horizontal direction. If the last horizontal jump is of the shape $n = 8k - 3$ or $n = 8k + 1$, then the total number of horizontal jumps is odd. This cannot happen. For the remaining case $n = 8k - 1$, we will use induction to prove that this case is possible.

Suppose that the last horizontal jump consists of $8k - 1$ steps. We will show that we can put pluses and minuses such that $\pm 1 \pm 3 \pm \cdots \pm (8k - 1) = 0$. For $k = 1$, we find $1 - 3 - 5 + 7 = 0$. Suppose that $\pm 1 \pm 3 \pm \cdots \pm (8j - 1) = 0$ for a certain $j \geqslant 1$. Then

$$\pm 1 \pm 3 \pm \cdots \pm (8j - 1) + (8j + 1) - (8j + 3) - (8j + 5) + (8j + 7) = 0 + 1 - 3 - 5 + 7 = 0$$

and we can choose pluses and minuses such that $\pm 1 \pm 3 \pm \cdots \pm (8(j + 1) - 1) = 0$. Now we proved that we can put pluses and minuses such that $\pm 1 \pm 3 \pm \cdots \pm (8k - 1) = 0$ for each integer k.

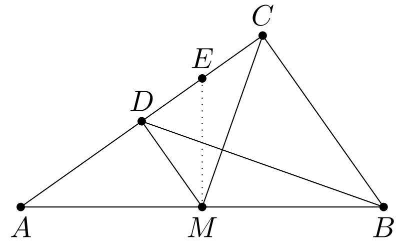
We conclude that there are two possibilities for the frog to end at the origin $(0, 0)$. The first is for $n = 8k - 1$: the second last jump consists of $8k - 2$ vertical steps, and the last jump consists of $8k - 1$ horizontal steps. The second is $n = 8k$, then the second last jump consists of $8k - 1$ horizontal steps and the last jump consists of $8k$ vertical steps. □

3. Version for klas 6

In the solution for klas 5 & klas 4 and below this problem is solved in two steps. In part (a), we consider a frog that is jumping only on the (horizontal) line. The frog is making a jump of size 1 to the left or right, a jump of size 2 to the left or right, a jump of size 3 to the left or right, et cetera. The solution shows for which n the frog can return to the number 0 after n jumps. This is then used in part (b) to show for which n a frog jumping both horizontally and vertically, can can return to the origin $(0, 0)$ after n jumps. □

4. Version for klas 4 & below

(a) We prove the similarity $\triangle CMD \sim \triangle ABC$. Since BC and MD are parallel, we find that $\angle ADM = \angle ACB = 90°$ and also $\angle AMD = \angle ABC$. It follows that $\triangle ABC \sim \triangle AMD$. Because $|AB| = 2|AM|$ we also have that $|AC| = 2|AD|$ and thus $|AD| = |DC|$. This implies the congruence $\triangle AMD \cong \triangle CMD$: both triangles have a right angle at D and the two adjacent sides have the same length. Now we have that $\triangle ABC \sim \triangle AMD \cong \triangle CMD$, and so it holds that $\triangle CMD \sim \triangle ABC$. □



(b) We prove that $\triangle CME \sim \triangle ABD$. From part (a) it follows that $\angle ECM = \angle DCM = \angle CAB = \angle DAB$, and also that

$$\frac{|EC|}{|CM|} = \frac{\frac{1}{2}|DC|}{|CM|} = \frac{\frac{1}{2}|CA|}{|AB|} = \frac{|DA|}{|AB|}.$$

This implies that $\triangle CME \sim \triangle ABD$: the triangles have one equal angle and the two adjacent sides have the same ratio. $\square$.

(c) Let F be the intersection of BD en CM. Since BD is perpendicular to CM we have that $\angle BFM = 90°$. So in the triangle $\triangle BFM$ we have that $\angle BMF + \angle FBM = 90°$. Because of the similar triangles in part (b) we have $\angle FBM = \angle ABD = \angle CME = \angle FME$. It follows that $\angle BMF + \angle FME = 90°$, hence EM is perpendicular to AB. $\square$

4. Version for klas 5 & klas 6

(a) In the solution for klas 4 and below this problem is solved in two steps. In part (a) it is proven that $\triangle CMD \sim \triangle ABC$. Then this is used in part (b) to show that $\triangle CME \sim \triangle ABD$. $\square$

(b) This is the same as the solution to part (c) of the solution for klas 4 and below. $\square$

5. Version for klas 4 & below

Suppose by contradiction that n is not prime. Now consider the greatest divisor $d < n$ of n. Then we can write n as de. Since n is not prime, we have $d > 1$ and hence also $e < n$. Now e must satisfy $e > 1$ and $e \leqslant d$ (because d is the greatest divisor satisfying $d < n$). Now $d + 1$ must be a divisor of $n + 1$. Moreover, $d + 1$ is a divisor of $(d + 1)e = de + e = n + e$. This means that $d + 1$ must also be a divisor of the difference $n + e - (n + 1) = e - 1$. This, however, is impossible, because $e - 1$ is a number between 1 and $d - 1$. Therefore, our assumption that n is not prime must be false, and n must actually be a prime number. $\square$

5. Version for klas 5 & klas 6

The solution for klas 5 & klas 6 is the same as the solution for klas 4 & below. $\square$