Final round
Dutch Mathematical Olympiad

Version klas 5

NEDERLANDSE WISKUNDE OLYMPIADE

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. Daan distributes the numbers 1 to 9 over the nine squares of a 3×3-table (each square receives exactly one number). Then, in each row, Daan circles the median number (the number that is neither the smallest nor the largest of the three). For example, if the numbers 8, 1, and 2 are in one row, he circles the number 2. He does the same for each column and each of the two diagonals. If a number is already circled, he does not circle it again.

⑧	1	②
7	⑥	③
9	⑤	4

He calls the result of this process a *median table*. Above, you can see a median table that has 5 circled numbers.

(a) What is the **smallest** possible number of circled numbers in a median table?
Prove that a smaller number is not possible and give an example in which a minimum number of numbers is circled.

(b) What is the **largest** possible number of circled numbers in a median table?
Prove that a larger number is not possible and give an example in which a maximum number of numbers is circled.

2. We consider number sequences $a_1, a_2, a_3, \ldots$ such that $a_{n+1} = \dfrac{a_n + a_1}{a_n + 1}$ holds for all $n \geqslant 1$.

(a) Suppose that $a_1 = -3$. Compute a_{2020}.

(b) Suppose that $a_1 = 2$. Prove that $\frac{4}{3} \leqslant a_n \leqslant \frac{3}{2}$ holds for all $n \geqslant 2$.

3. Given is a parallelogram $ABCD$ with $\angle A < 90°$ and $|AB| < |BC|$. The angular bisector of angle A intersects side BC in M and intersects the extension of DC in N. Point O is the centre of the circle through M, C, and N.
Prove that $\angle OBC = \angle ODC$.
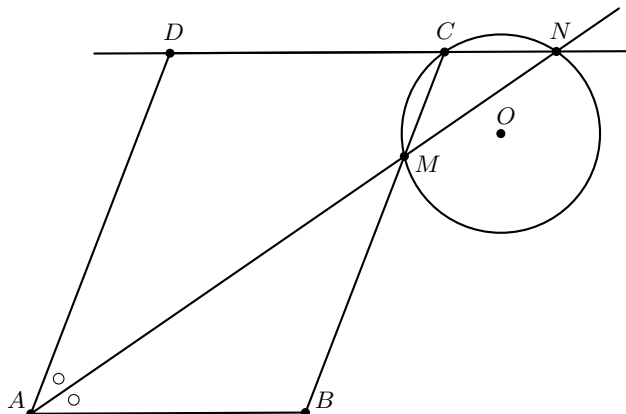
4. Determine all pairs of integers (x, y) such that $2xy$ is a perfect square and $x^2 + y^2$ is a prime number.

5. Sabine has a very large collection of shells. She decides to give part of her collection to her sister.

On the first day, she lines up all her shells. She takes the shells that are in a position that is a perfect square (the first, fourth, ninth, sixteenth, etc. shell), and gives them to her sister. On the second day, she lines up her remaining shells. Again, she takes the shells that are in a position that is a perfect square, and gives them to her sister. She repeats this process every day.

The 27th day is the first day that she ends up with fewer than 1000 shells. The 28th day she ends up with a number of shells that is a perfect square for the tenth time.
What are the possible numbers of shells that Sabine could have had in the very beginning?