Finale
Dutch Mathematical Olympiad

Version klas 6

NEDERLANDSE WISKUNDE OLYMPIADE

Friday 13 September 2019
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded for partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

1. A *complete number* is a 9 digit number that contains each of the digits 1 to 9 exactly once. The *difference number* of a number N is the number you get by taking the differences of consecutive digits in N and then stringing these digits together. For instance, the difference number of 25143 is equal to 3431. The complete number 124356879 has the additional property that its difference number, 12121212, consists of digits alternating between 1 and 2.
Determine all a with $3 \leqslant a \leqslant 9$ for which there exists a complete number N with the additional property that the digits of its difference number alternate between 1 and a.

2. There are n guests at a party. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is *not* friends with two other guests, those two other guests cannot be friends with each other either.
What are the possible values of n?

3. Points A, B, and C lie on a circle with centre M. The reflection of point M in the line AB lies inside triangle ABC and is the intersection of the angular bisectors of angles A and B. (The angular bisector of an angle is the line that divides the angle into two equal angles.) Line AM intersects the circle again in point D.
Show that $|CA| \cdot |CD| = |AB| \cdot |AM|$.

4. The sequence of Fibonacci numbers $F_0, F_1, F_2, \ldots$ is defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all $n \geqslant 0$. For example, we have

$$F_2 = F_0 + F_1 = 2, \quad F_3 = F_1 + F_2 = 3, \quad F_4 = F_2 + F_3 = 5, \quad \text{and} \quad F_5 = F_3 + F_4 = 8.$$

The sequence $a_0, a_1, a_2, \ldots$ is defined by

$$a_n = \frac{1}{F_n F_{n+2}} \quad \text{for all } n \geqslant 0.$$

Prove that for all $m \geqslant 0$ we have:

$$a_0 + a_1 + a_2 + \cdots + a_m < 1.$$

5. SEE NEXT PAGE FOR PROBLEM 5.

5. Thomas and Nils are playing a game. They have a number of cards, numbered 1, 2, 3, et cetera. At the start, all cards are lying face up on the table. They take alternate turns. The person whose turn it is, chooses a card that is still lying on the table and decides to either keep the card himself or to give it to the other player. When all cards are gone, each of them calculates the sum of the numbers on his own cards. If the difference between these two outcomes is divisible by 3, then Thomas wins. If not, then Nils wins.

(a) Suppose they are playing with 2018 cards (numbered from 1 to 2018) and that Thomas starts. Prove that Nils can play in such a way that he will win the game with certainty.

(b) Suppose they are playing with 2020 cards (numbered from 1 to 2020) and that Nils starts. Which of the two players can play in such a way that he wins with certainty?