Final Round
Dutch Mathematical Olympiad

Friday 16 September 2016

Solutions

1. (a) For brevity, we will say that a number encircled in red is a red number, and similarly for blue. So some numbers could be both red and blue. Since there are 999 numbers written on the pavement, of which 500 are red and 500 are blue, we have at least one bicoloured number by the pigeonhole principle. Consider such a bicoloured number and suppose it is the number k. From left to right, the red numbers form the sequence $1, 2, \ldots, 500$. Hence, to the left of the bicoloured number we have the red numbers 1 to $k-1$, and to the right we have the red numbers $k+1$ to 500. The blue numbers are written in the opposite order: from left to right they form the sequence $500, 499, \ldots, 1$. To the left of the bicoloured number we therefore have the blue numbers $k+1$ to 500, and to the right we have the blue numbers 1 to $k-1$.

We count how many numbers there are on each side of the bicoloured number. On the left we have the red numbers 1 to $k-1$ and the blue numbers $k+1$ to 500. Hence, there are at least 499 distinct numbers on the left. On the right we have the red numbers $k+1$ to 500 and the blue numbers 1 to $k-1$. Again at least 499 distinct numbers. Since there are only $999 = 499 + 1 + 499$ numbers on the pavement, we have already considered all numbers on the pavement. We conclude that there are precisely 499 numbers on each side of the bicoloured number. The bicoloured number is therefore precisely in the middle of the sequence.

(b) Just as in part (a), the pigeonhole principle yields that at least one number is bicoloured, i.e. both red and green. If more than one number is bicoloured, one of them is not exactly in the middle of the sequence and we are done. Therefore, it suffices to examine the case where there is exactly one bicoloured number. Let this number be equal to k.

Again, we count how many numbers there are on each side of the bicoloured number. On the left we have the red numbers 1 to $k-1$ and the green numbers 1 to $k-1$. Since none of these numbers is bicoloured, there are at least $2 \cdot (k-1)$ distinct numbers on the left. On the right we have the red numbers $k+1$ to 500 and the green numbers $k+1$ to 500. Since none of these numbers is bicoloured, we have at least $2 \cdot (500-k)$ distinct numbers on the right. Since $2 \cdot (k-1) + 1 + 2 \cdot (500-k) = 999$, we have already counted all numbers.

On the left of the bicoloured number we therefore have exactly $2 \cdot (k-1)$ numbers, and on the right we have exactly $2 \cdot (500-k)$ numbers. Since $2 \cdot (k-1)$ is an *even* number, it is unequal to 499. We conclude that the bicoloured number is not exactly in the middle of the sequence.

2. Suppose that our sequence has x ones, y minus ones (and hence $2n - x - y$ zeroes). We calculate the sum product value of the sequence (as an expression in x and y).

In the sum product value, six different types of terms occur: $1 \cdot 1$, $1 \cdot -1$, $-1 \cdot -1$, $1 \cdot 0$, $-1 \cdot 0$, and $0 \cdot 0$. Only the first three types contribute since the other types are equal to 0.

The number of terms of the type $1 \cdot 1 = 1$ equals the number of ways to select two out of x ones. This can be done in $\frac{x(x-1)}{2}$ ways: there are x options for the first 1, and then $x-1$ options for the second 1. Since the order in which we select the two ones does not matter, we effectively count each possible pair twice.

Similarly, the number of terms of the type $-1 \cdot -1 = 1$ is equal to $\frac{y(y-1)}{2}$.

The number of terms of the type $1 \cdot -1 = -1$ is equal to xy, since there are x options for choosing a 1 and, independently, there are y options for choosing a -1.

In total, we obtain a sum product value of

$$S = \frac{x(x-1)}{2} \cdot 1 + \frac{y(y-1)}{2} \cdot 1 + xy \cdot -1 = \frac{x^2 - 2xy + y^2 - x - y}{2} = \frac{(x-y)^2 - (x+y)}{2}.$$

Since $(x-y)^2 \geqslant 0$ (squares are non-negative) and $-(x+y) \geqslant -2n$ (there are only $2n$ numbers in the sequence), we see that $S \geqslant \frac{0-2n}{2} = -n$. So the sum product value cannot be smaller than $-n$. If we now choose $x = y = n$, then $(x-y)^2 = 0$ and $-x-y = -2n$, which imply a sum product value of exactly $\frac{0-2n}{2} = -n$. Therefore, the smallest possible sum product value is $-n$.

3. The problem is symmetric in a, b, and c. That is, if we consistently swap a and b, or a and c, or b and c, then the conditions on (a,b,c) do not change. We will therefore consider solutions for which $a \leqslant b \leqslant c$. The remaining solutions are then found by permuting the values of a, b, and c.

Since a and b are positive, we see that $a+b+c > c$. Since c is largest among the three numbers, we also have $a+b+c \leqslant 3c$. Since we are given that $a+b+c$ is a multiple of c, we are left with two possibilities: $a+b+c = 2c$ or $a+b+c = 3c$. We consider both cases separately. If $a+b+c = 3c$, then a, b, and c must all be equal, because otherwise, the fact that $a, b \leqslant c$ implies that $a+b+c < 3c$. This means that $\gcd(b,c) = \gcd(c,c) = c$. Since $\gcd(b,c)$ must be equal to 1, we find $(a,b,c) = (1,1,1)$. This is indeed a solution, since $\gcd(1,1) = 1$ and 1 is a divisor of $1+1+1$.

If $a+b+c = 2c$, then $c = a+b$. We know that b must be a divisor of $a+b+c = 2a+2b$. Since $a > 0$, we have $2a+2b > 2b$. Since $b \geqslant a$, we also have $2a+2b \leqslant 4b$. Therefore, since $2a+2b$ must be a multiple of b, there are only two possibilities: $2a+2b = 3b$ or $2a+2b = 4b$. Again, we consider these cases separately.

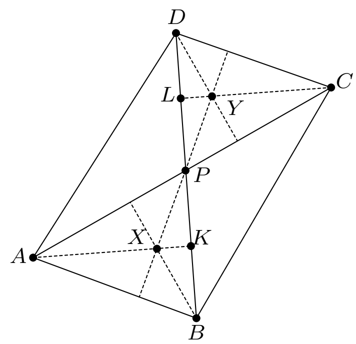
If $2a+2b = 3b$, then $b = 2a$. Similarly to the first case, we find that $a = \gcd(a,2a) = \gcd(a,b) = 1$. Therefore, $b = 2$ and $c = a+b = 3$. The resulting triple $(a,b,c) = (1,2,3)$ is indeed a solution, since $\gcd(1,2) = \gcd(1,3) = \gcd(2,3) = 1$ and $1+2+3 = 6$ is divisible by 1, 2, and 3.

If $2a+2b = 4b$, then $a = b$. Again we see that $a = \gcd(a,a) = \gcd(a,b) = 1$. From $b = a = 1$ it follows that $c = a+b = 2$ and hence $(a,b,c) = (1,1,2)$. This is indeed a solution since $\gcd(1,1) = \gcd(1,2) = 1$ and $1+1+2 = 4$ is divisible by 1 and 2.

We conclude that the solutions for which $a \leqslant b \leqslant c$ holds are: $(a,b,c) = (1,1,1)$, $(a,b,c) = (1,1,2)$, and $(a,b,c) = (1,2,3)$. Permuting the values of a, b and c, we obtain a total of ten solutions (a,b,c):

$$(1,1,1), \quad (1,1,2),\ (1,2,1),\ (2,1,1), \quad (1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1).$$

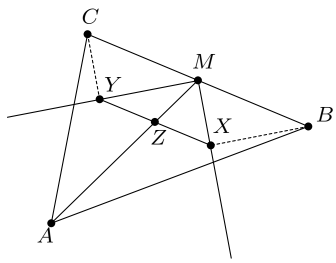
4. **[Version klas 5 & klas 4 and below]** Let K be the intersection of AX and BD, and let L be the intersection of CY and BD. Consider the triangles PLY and PKX. The angles $\angle PLY$ and $\angle PKX$ are both right angles. The angles $\angle YPL$ and $\angle XPK$ are opposite angles and therefore equal. Since $|PX| = |PY|$, we see that triangles PKX and PLY are congruent (SAA). Hence, $|PK| = |PL|$.



Now consider triangles PAK and PCL. Angles $\angle AKP$ and $\angle CLP$ are both right angles. Angles $\angle KPA$ and $\angle LPC$ are opposite angles, hence equal. We have already shown that $|PK| = |PL|$. Therefore, triangles PAK and PCL are congruent (ASA). From this, we conclude that $|AP| = |PC|$.

In a similar fashion, we may deduce that $|BP| = |DP|$. The two diagonals of $ABCD$ bisect each other, hence $ABCD$ is a parallelogram.

4. **[Version klas 6]** We start by observing that $\angle CMY = \frac{1}{2}\angle CMA = \frac{1}{2}(180^\circ - \angle AMB) = 90^\circ - \angle XMB$. Since $\angle MXB = 90^\circ$, and the angles of triangle BMX sum to 180°, we see that $\angle CMY = 90^\circ - \angle XMB = \angle MBX$.



Looking at triangles CMY and MBX, we observe that $\angle CMY = \angle MBX$, $\angle MYC = 90^\circ = \angle BXM$, and $|CM| = |MB|$. The two triangles are therefore congruent (SAA). In particular, we obtain the equalities $|MX| = |CY|$ and $|MY| = |BX|$.

Now consider triangle XYM. We already know that $|MY| = |BX|$ and that $\angle YMX = \angle YMA + \angle AMX = \frac{1}{2}\angle CMA + \frac{1}{2}\angle AMB = \frac{1}{2} \cdot 180^\circ = 90^\circ$. Since triangles XYM and MBX also share the side MX, they are congruent (SAS).

In particular, we see that $\angle MXY = \angle XMB$. Since MX is the angle bisector of $\angle AMB$, we have $\angle XMB = \angle AMX$. This implies that triangle MXZ has two equal angles and is therefore an isosceles triangle with vertex angle Z. We conclude that $|MZ| = |XZ|$.

In a similar manner, we see that triangles XYM and CMY are congruent, and find that $\angle XYM = \angle CMY = \angle YMA$. Triangle MYZ is therefore isosceles with vertex angle Z. This implies that $|YZ| = |MZ|$. Together with $|MZ| = |XZ|$, this concludes the proof.

5. Suppose that not all numbers are coloured blue. Then, there must be a number k that is not blue. We will use this to derive a contradiction.

Without loss of generality, we may assume that k is coloured red. Since all *odd* numbers are blue, k must be *even*, say $k = 2m$ for some integer $m \geqslant 1$. From the second requirement, it follows that $8m$ is red as well. From the third requirement, it now follows that at least one of the numbers $8m + 2$ and $8m + 4$ is red. However, $2m + 1$ is odd and therefore blue, which by the second requirement implies that $8m + 4 = 4 \cdot (2m + 1)$ is blue as well. So $8m + 2$ must be red.

By the third requirement, $8m - 2$ must be the same colour as $8m$ or $8m + 2$. Since both $8m$ and $8m + 2$ are red, this implies that $8m - 2$ must be red as well. Since $8m$ and $8m - 2$ are red, this implies (again by the third requirement) that $8m - 4$ is also red. The second requirement now implies that $(8m - 4)/4 = 2m - 1$ is also red. But that is impossible since $2m - 1$ is odd, and therefore blue.

We conclude that the assumption that not all numbers are blue leads to a contradiction. Therefore, all numbers must be blue.