Second round Dutch Mathematical Olympiad

Friday 15 March 2024

Solutions

B-problems

B1. 19 An envelope with width 30 cannot fit on any other envelope, and an envelope with height 20 cannot fit on any other envelope either. So for the envelopes with dimensions 21×20, 22×20, $\ldots$, 30×20, 30×19, $\ldots$, 30×11, Milou has to make 19 different piles, with these envelopes at the bottom. We can see that the rest of the envelopes can all be put on top of this as follows. For each envelope, calculate the difference between width w and height h. This difference is equal to a number from 1 through 19, and for all 19 envelopes Milou has already placed, every difference occurs exactly once. All envelopes for which $w - h$ is equal, fit on the same pile. For example, the pile for $w - h = 5$ looks like this from bottom to top: 25×20, 24×19, 23×18, 22×17 and 21×16. So Milou can put all the envelopes in 19 piles and less piles are impossible.

B2. 712 Let us systematically count how many digits the page numbers contain, not yet taking into account the sheet that is torn out. Pages 1 through 9 have one digit each, so a total of nine digits. Pages 10 through 99 are 90 pages with two digits each, that is $90 \cdot 2 = 180$ digits in total. So for pages 1 through 99, this adds up to 189 digits.

Call the total number of digits on the front and back of the torn-out sheet x. So originally (before tearing out) the book had $2024 + x$ digits. So the number of digits on pages 100 through n, on the one hand, equals $(2024 + x) - 189 = 1835 + x$. On the other hand, since each of the pages 100 through n has three digits, that number of digits is equal to $3(n - 99)$. Since $2 \leqslant x \leqslant 6$, this number is equal to one of the numbers 1837 through 1841. This number must also be divisible by 3, so only 1839 is possible. We find that $3(n - 99) = 1839$ and so $n = 712$.

B3. 16 Denote the number of students in the class by n. Since there are at most 25 pupils in the class, each pupil is at least 4% of the total. The 94% pupils who live more than 1 km from school must be $n - 1$ pupils: after all, they are not all n pupils, and $n - 2$ pupils can only be at most 92% of the total. Hence, one student of the class is 6% of the class, rounded.

We found that one student corresponds with a percentage between 5.5 and 6.5. This means that 31% corresponds to 5 students, because 4 students give at most $4 \cdot 6.5 = 26$ percent and 6 students give at least $6 \cdot 5.5 = 33$ percent.

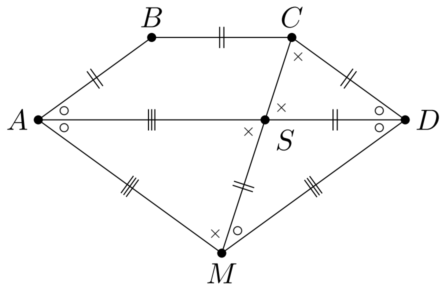
Hence, five students corresponds to a percentage between 30.5 and 31.5. That means that 15 students contribute to at most $\frac{15}{5} \cdot 31.5 < 95$ percent and 17 students to at least $\frac{17}{5} \cdot 30.5 > 17 \cdot 6 = 102$ percent. Thus the number of students is equal to 16.

B4. $\frac{4}{7}$ We first look at some smaller cases with only 0, 1 or 2 times the digit combination 84:

$$\frac{4}{7} = \frac{4}{7}, \quad \frac{484}{847} = \frac{4 \cdot 121}{7 \cdot 121} = \frac{4}{7}, \quad \frac{48484}{84847} = \frac{4 \cdot 12121}{7 \cdot 12121} = \frac{4}{7}.$$

Also, for our very large number A, we immediately see that $A = 484\ldots84 = 4 \cdot 121\ldots21$, while we can see B equals $7 \cdot 121\ldots21$. It follows then that $\frac{A}{B} = \frac{4}{7}$.

B5. 24 Because of symmetry, the line segments from M to the vertices divide the decagon into ten congruent triangles. The angle at M of such a triangle is equal to $\frac{360°}{10} = 36°$ (represented in the figure by $\circ$) and the other two angles which are the same size, are $\frac{180°-36°}{2} = 72°$ (represented in the figure by $\times$). It follows that the angles of the original decagon are each $2 \cdot 72° = 144°$. For the rest of the solution, we are not going to look at the whole decagon again, only at the "diamond" given by the pentagon $ABCDM$. See the picture to the right.



First, we look at quadrilateral $ABCD$. The sum of the four angles is 360°. The angles at B and C are 144° and thus the angles at A and D, which are equal because of the symmetry of the decagon, are 36°. This means that the line segment AD divides the angles $\angle MAB$ and $\angle MDC$ into two equal pieces of 36°. This also implies $\angle MAS = 36°$ and $\angle MDS = 36°$.

Now consider triangle SDC. Since the sum of the angles is equal to 180°, we find that angle $\angle CSD = 180° - \angle CDS - \angle MCD = 180° - 36° - 72° = 72°$ and the triangle is therefore isosceles. It follows that $|SD| = |CD| = 12$, the side of the decagon (marked with two dashes in the figure).

Then consider triangle MAS. Because of opposite angles, $\angle MSA = 72°$. We already saw that $\angle MAS = 36°$. Because of the angle sum of 180° in triangle MAS, we have $\angle AMS = 72°$ and so this triangle too is isosceles. It follows that $|SA| = |MA|$, the radius of the decagon (marked with three dashes in the figure).

We consider the triangle MDS. We already saw that $\angle MDS = 36°$, and at the beginning we also saw that $\angle DMS = 36°$. This is again an isosceles triangle, so $|MS| = |DS|$ and we already saw that this is equal to 12.

Finally, we can calculate the difference between the perimeter of quadrilateral $ABCD$ and the perimeter of triangle DMS:

$$\begin{aligned}
&(|AB| + |BC| + |CD| + |DA|) - (|DM| + |MS| + |SD|) \\
&= (12 + 12 + 12 + (12 + |SA|)) - (|DM| + 12 + 12) \\
&= 24 + |MA| - |MD| \\
&= 24.
\end{aligned}$$

C-problems

C1. Let's start with $n = 9$. Then the grasshopper can jump as follows: $1-3-6-8-5-2-4-7-9$. You can see that the grasshopper jumps to larger numbers first, then to smaller numbers each time, and then to larger numbers again. This 'zigzag' pattern can be used in general.

We distinguish different cases for n based on the remainder of n when dividing by 3. If $n = 3k$, then the grasshopper can jump as follows:

$$1; \quad 3, 6, \ldots, 3(k-1); \quad 3(k-1)+2, 3(k-2)+2, \ldots, 5, 2; \quad 4, 7, \ldots, 3(k-1)+1; \quad 3k.$$

The grasshopper jumps over triples on the way out, over triples plus 2 on the way back, and then over triples plus 1 on the second way out.

If $n = 3k+1$, then the grasshopper can jump as follows:

$$1; \quad 3, 6, \ldots, 3k; \quad 3k-2, 3(k-1)-2, \ldots, 7, 4; \quad 2, 5, 8, \ldots, 3(k-1)+2; \quad 3k+1.$$

Finally, for $n = 3k+2$, we find that the grasshopper can jump as follows:

$$1, 3, 6, 4; \quad 2, 5, 8, \ldots, 3(k-1)+2; \quad 3k+1, 3(k-1)+1, \ldots, 7; \quad 9, \ldots, 3k; \quad 3k+2.$$

Because $n \geqslant 9$ we have that $k \geqslant 3$. In the solutions above for $k \geqslant 3$ we see that every leg of the 'zigzag' is nonempty, so a solution exists.

There are other ways to solve this problem. For example, you can also give a solution for $n = 9, 10, 11, 12$ *and* 13 *and then make a solution for* $n + 5$ *by starting with a solution for* n *and adding* $n + 2, n + 4, n + 1, n + 3, n + 5$ *to it.*

C2. (a) Add up the positions of the characters 'L', where the leftmost character in the word has position 1 and the rightmost character has position n. We call this number the *L-sum* of a word. For each word, the L-sum is a non-negative integer. Furthermore, for every move Eva makes, the L-sum becomes one lower. Indeed, when switching 'L' and 'R', the position of the 'L' that Eva switches becomes one lower. Therefore, since the L-sum cannot become negative, Eva can always do only a finite number of turns.

(b) Of all the possible words of length n that Eva considers, the L-sum is the largest with the word 'RR...RL...LL', where all ℓ characters 'L' are on the right side of the word. On the contrary, the L-sum is smallest for the word 'LL...LRR...R', where all ℓ characters 'L' are on the left side of the word. In this word, Eva cannot do any more turns, because there is nowhere an 'L' directly to the right of an 'R'.

The left-most 'L' in 'RR...RL...LL' and the left-most 'L' in 'L...LRR...R' differ $n - \ell$ from each other in position. The same is true for all subsequent characters 'L', from left to right. Thus, the difference in L-sum between these two words is $\ell(n - \ell)$. We already saw that the L-sum of a word becomes exactly one smaller at each turn: an upper bound on the maximum number of turns is thus $\ell(n - \ell)$.

Eva can also actually do $\ell(n - \ell)$ turns if she starts with the word 'RR...RL...LL'. For the first $n - \ell$ turns, she uses only the leftmost 'L', and the result is the word 'LRR...RL...LL' with $\ell - 1$ times an 'L' on the right side. Next, she chooses the second 'L' from the left, and in $n - \ell$ turns she makes the word 'LLRR...RL...LL' with $\ell - 2$ times an 'L' on the right side. Eva does this with all ℓ the characters 'L'. In total, she can take $\ell(n - \ell)$ turns before she ends with 'L...LRR...R'.

(c) In the previous part of the problem, we already saw that Eva can do at most $\ell(n - \ell)$ turns. Consider the function $f(\ell) = \ell(n - \ell)$. This is a quadratic function with zeros at $\ell = 0$ and $\ell = n$. So the maximum is at $\ell = \frac{1}{2}n$. If n is even, then Eva can do as many turns as possible at $\ell = \frac{n}{2}$. (The number of turns is then $f\left(\frac{n}{2}\right) = \frac{1}{4}n^2$.) If n is odd, the maximum of this function is not at an integer value of ℓ and we see that Eva can do as many turns as possible at $\ell = \frac{n-1}{2}$ and $\ell = \frac{n+1}{2}$. (The number of turns is then $f\left(\frac{n-1}{2}\right) = f\left(\frac{n+1}{2}\right) = \frac{1}{4}(n^2 - 1)$.)