Second round
Dutch Mathematical Olympiad

Friday 15 March 2024

- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- You are not allowed to use a calculators or other electronic device, nor a formula sheet. You can only use a pen, compass, ruler, set square, and of course your mental skills.
- Good luck!

B-problems

In B-problems, you only have to give the answer (e.g. a number). An explanation is not required. You will get 4 points for a correct answer and 0 points for a wrong or incomplete answer. So work calmly and accurately, as a small miscalculation may result in your answer being wrong.
REMEMBER: give your answers in exact and simplified form such as $\frac{11}{81}$ or $2+\frac{1}{2}\sqrt{5}$ or $\frac{1}{4}\pi+1$ or 3^{100}.

B1. Milou has 100 long envelopes of different sizes. Each envelope has a width equal to one of the integers $21,\ldots,30$ and a height equal to one of the integers $11,\ldots,20$ and each combination occurs exactly once. Milou wants to organise the envelopes into piles. An envelope may only be placed on top of another envelope if both its width and height are smaller than that of the envelope she is placing it on. So the size 26×15 envelope is allowed on top of the 29×17 envelope, but not on the 29×15 envelope or the 26×17 envelope.
What is the smallest number of piles into which Milou can organise the envelopes?

B2. Charlie has a thick book of n pages in which, after opening, the first page on the right shows the page number 1. Then all the pages are numbered continuously so that if you open the book just anywhere, you see an even page number on the left and an odd page number on the right. The very last page before the back cover has an even number. Unfortunately, someone tore a whole sheet of paper out of the book. Charlie counts the total number of <u>digits</u> of the page numbers on the $n-2$ remaining pages and arrives at a total of 2024 digits. (*Note that this is <u>not</u> the sum of the page numbers.*)
What is n?

B3. At Pythagoras Lyceum, all classes have a maximum of 25 students. A survey is held in one of the classes about all sorts of things and everyone fills in all the questions. In this class, it turns out that 31% of the students have a black bicycle and 94% live more than 1 km away from the school. These percentages have been rounded to the closest integer.
How many students are in that class?

B4. We define the number A as the digit 4 followed by 2024 times the digit combination 84, and the number B by 2024 times the digit combination 84 followed by a 7. So, to illustrate, we have

$$A = 4\,\underbrace{84\ldots84}_{\text{4048 digits}} \quad \text{and} \quad B = \underbrace{84\ldots84}_{\text{4048 digits}}\,7.$$

Simplify the fraction $\frac{A}{B}$ as much as possible.

B5. Of a regular decagon $ABCDEFGHIJ$ with sides of length 12, M is the centre. Let S be the intersection of MC and AD.
Calculate the difference between the perimeter of quadrilateral $ABCD$ and the perimeter of triangle DMS.

C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning that shows the correctness of your answer. A correct and well-explained answer is awarded 10 points. Partial solutions may also be worth some points. Therefore, write everything down clearly and hand in your drafts.
ATTENTION: Use separate sheets of paper for each C-problem and also hand in the drafts for each problem separately.

C1. Let n be a positive integer. A grasshopper stands on the number line at the number 1 and may make either a jump of length 2 or of length 3 each time. Each time, the grasshopper must land on an integer from 1 through n where the grasshopper has not been before. The grasshopper would like to visit all integers from 1 through n exactly once and land on the number n.
Prove that this can be done for all $n \geqslant 9$.

C2. Eva looks at "words" consisting of n characters, each equal to 'L' or 'R'. In one turn, Eva may replace 'RL' anywhere in the word with 'LR'. For example, in two turns, she takes the word 'LR<u>RL</u>R<u>RL</u>R' to the word 'LR<u>LR</u>R<u>LR</u>R'. If there is no 'L' immediately to the right of an 'R', Eve cannot make a turn.

(a) Eva has such a word of length n. Prove that Eva can only make a finite number of turns.

(b) Given $n > 1$ and ℓ with $0 < \ell < n$. For every word of length n with exactly ℓ times an 'L' Eva writes down how many turns she can take at most. What is the biggest number she wrote down? (Give your answer in terms of n and ℓ.)
Prove that your answer is correct. This means: give a word of length n with exactly ℓ times the character 'L' for which the maximum number of turns is achieved and prove that the maximum number of turns for all other words of length n with exactly ℓ times the character 'L' cannot be greater.

(c) Let $n \geqslant 2$. For every word of length n (with 1 or more times an 'L' and 1 or more times an 'R') Eva writes down how many turns she can take at most. How many characters 'L' contains the word (or words) for which she has written down the biggest number? (Give your answer in terms of n.)