Second round
Dutch Mathematical Olympiad

Friday 12 March 2021

- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- Formula sheets and calculators are not allowed. You can only use a pen, compass, ruler, set square, and of course your mental skills.
- Good luck!

B-problems

For the B-problems you only have to give the answer (for example, a number). No explanation is required. A correct answer is awarded 4 points. For a wrong or incomplete answer no points are given. Please work very accurately: a minor error in a calculation may result in a wrong answer.
NOTE: All answers should be given in exact and reduced form, like $\frac{11}{81}$, or $2+\frac{1}{2}\sqrt{5}$, or $\frac{1}{4}\pi+1$, or 3^{100}.

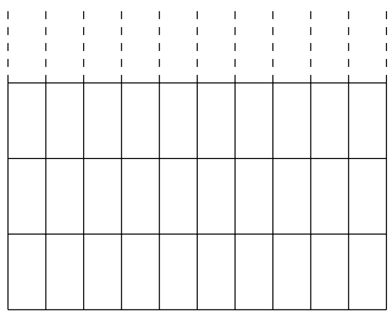
B1. Peter gets bored during the lockdown, so he decides to write numbers the whole day. He makes a sequence of numbers starting with 0, 1 and -1, and then going on indefinitely. On the next line he writes the same sequence of numbers, but shifted one place to the right. On the third line he writes again the same sequence of numbers, shifted another place to the right. He adds all three numbers standing in a vertical column. (He skips the first two places so he starts with $-1+1+0$.) The answer for every column is the next multiple of three. Peter's paper hence looks like this:

$$\begin{array}{cccccccc} & 0 & 1 & -1 & \dots & \dots & \dots & \dots \\ & & 0 & 1 & -1 & \dots & \dots & \dots \\ + & & & 0 & 1 & -1 & \dots & \dots \\ \hline & & & 0 & 3 & 6 & 9 & 12 \end{array}$$

The first number in the uppermost sequence is 0, the second number is 1, the third number is -1, etcetera. Determine the 2021st number in the uppermost sequence.

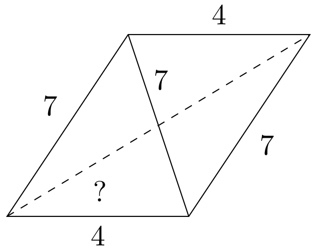
B2. An integer n is a *combi number* if each pair of distinct digits from the set of all possible digits 0 to 9 appear at least once in the number as neighbouring digits. For example, in a combi number the digits 3 and 5 have to appear somewhere next to each other. It does not matter whether they appear in the order 35 or 53. We take the convention that a combi number never starts with the digit 0.
What is the smallest possible number of digits of a combi number?

B3. A big rectangle is divided in small rectangles that are twice as high as they are wide. The rectangle is 10 of these small rectangles wide, as in the figure on the right. In this figure you can see some squares of different sizes.
How many small rectangles high is the figure if we can find exactly 345 squares in it?

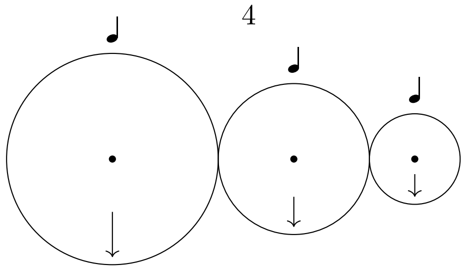


PLEASE CONTINUE ON THE OTHER SIDE

B4. A parallelogram has two sides of length 4 and two sides of length 7. Also, one of the diagonals has length 7. (Attention: the picture has not been drawn to scale.)
What is the length of the other diagonal?



B5. Three wheels are pushed together so they don't slip if we turn them. The circumferences of the wheels are 14, 10, and 6 cm, respectively. On each wheel an arrow is drawn, pointing downwards. Someone turns the big wheel and the other wheels turn with it. This stops at the first moment all arrows point downwards again. Every time one of the arrows is pointing up, a whistle sounds. If two or three arrows point up at the same time, only one whistle sounds.
How many whistles sound in total?



C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning that shows the correctness of your answer. A correct and well-explained answer is awarded 10 points. Partial solutions may also be worth some points. Therefore, write everything down clearly and hand in your drafts.
ATTENTION: Use separate sheets of paper for each C-problem and also hand in the drafts for each problem separately.

C1. Around a round table $n \geqslant 3$ players are sitting. The game leader divides n coins among the players, in such a way that not everyone gets exactly one coin. Any player can see the number of coins of each other player.
Every 10 seconds, the game leader rings a bell. At that moment, each player looks how many coins their two neighbours have. Then they all do the following at the same time:

- If a player has more coins than at least one of their neighbours, the player gives away exactly one coin. They give this coin to the neighbour with the smallest number of coins. If both of their neighbours have the same number of coins, they give the coin to the neighbour on the left.
- If a player does not have more coins than at least one of their neighbours, the player does nothing and waits for the next round.

The game ends if everyone has exactly one coin.

(a) For each $n \geqslant 3$, find a distribution of the coins at the start such that the game will never stop (and prove that the game does not stop for your starting distribution).
(b) For each $n \geqslant 4$, find a distribution of the coins at the start of the game such that the game will stop (and prove that the game stops for your starting distribution).

C2. We consider a triangle ABC and a point D on the extended line segment AB on the side of B. The point E lies on side AC such that the angles $\angle DBC$ and $\angle DEC$ are equal. The intersection of DE and BC is F. Suppose that $|BF| = 2$, $|BD| = 3$, $|AE| = 4$, and $|AB| = 5$. (Attention: the picture has not been drawn to scale.)

(a) Prove that triangles $\triangle ABC$ and $\triangle AED$ are similar.
(b) Determine $|CF|$.