Second round
Dutch Mathematical Olympiad

Friday 15 March 2019

- Time available: 2.5 hours.
- The competition consists of five B-problems and two C-problems.
- Formula sheets and calculators are not allowed. You can only use a pen, compass, ruler, set square, and of course your mental skills.
- Good luck!

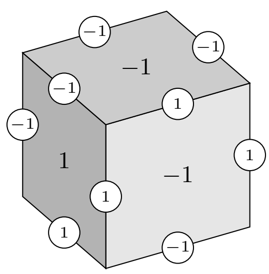
B-problems

For the B-problems you only have to give the answer (for example, a number). No explanation is required. A correct answer is awarded 4 points. For a wrong or incomplete answer no points are given. Please work very accurately: a minor error in a calculation may result in a wrong answer.
NOTE: All answers should be given in exact and reduced form, like $\frac{11}{81}$, or $2+\frac{1}{2}\sqrt{5}$, or $\frac{1}{4}\pi+1$, or 3^{100}.

B1. After breakfast, the sisters Anna and Birgit depart for school, each going to a different school. Their house is next to a bicycle path running between the two schools. Anna is cycling with a constant speed of 12 km per hour and Birgit is walking in the opposite direction with a constant speed of 4 km per hour. They depart at the same time. Shortly after their departure, mother notes that the girls have forgotten their lunch and decides to go after them. Exactly 10 minutes after Anna and Birgit have left, mother departs on her electric bike. First, she catches up with Anna. She hands her a lunch box, immediately turns around, and goes after Birgit. When she catches up with Birgit, she hands her a lunch box and immediately rides back home. Mother always rides at a constant speed of 24 km per hour.
How many minutes after the departure of Anna and Birgit does mother return home?

B2. In a tall hat there are one hundred notes, numbered from 1 to 100. You want to have three notes with the property that each of the three numbers is smaller than the sum of the other two. For example, the three notes numbered 10, 15, and 20 would be suitable (as $10 < 15+20$, $15 < 10+20$, and $20 < 10+15$), but the notes numbered 3, 4, and 7 would not (as 7 is not smaller than $3+4$). You may (without looking at the numbers on the notes) take some notes from the hat.
What is the smallest number of notes you have to take to be sure to have three notes that meet your wish?

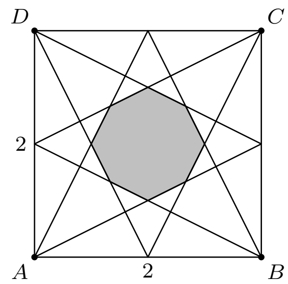
B3. On each of the twelve edges of a cube we write the number 1 or -1. For each face of the cube, we multiply the four numbers on the edges of this face and write the outcome on this face. Finally, we add the eighteen numbers that we wrote down.
What is the smallest (most negative) result we can get?
In the figure you see an example of such a cube. You cannot see the numbers on the back of the cube.



PLEASE CONTINUE ON THE OTHER SIDE

B4. If you try to divide the number 19 by 5, you will get a remainder. The number 5 fits 3 times in 19 and you will be left with 4 as remainder. There are two positive integers n having the following property: if you divide n^2 by $2n+1$, you will get a remainder of 1000.
What are these two integers?

B5. In a square $ABCD$ of side length 2 we draw lines from each vertex to the midpoints of the two opposite sides. For example, we connect A to the midpoint of BC and to the midpoint of CD. The eight resulting lines together bound an octagon inside the square (see figure).
What is the area of this octagon?



C-problems

For the C-problems not only the answer is important; you also have to write down a clear reasoning that shows the correctness of your answer. A correct and well-explained answer is awarded 10 points. Partial solutions may also be worth some points. Therefore, write everything down clearly and hand in your drafts.
ATTENTION: Use separate sheets of paper for each C-problem and also hand in the drafts for each problem separately.

C1. We consider sequences $a_1, a_2, \ldots, a_n$ consisting of n integers. For given $k \leqslant n$, we can partition the numbers of the sequence into k groups as follows: a_1 goes in the first group, a_2 in the second group, and so on until a_k which goes in the k-th group. Then a_{k+1} goes in the first group again, a_{k+2} in the second group, and so on. The sequence is called *k-composite* if this partition has the property that the sums of the numbers in the k groups are equal.

The sequence $1, 2, 3, 4, -2, 6, 13, 12, 17, 8$, for instance, is 4-composite as

$$1 + (-2) + 17 \;=\; 2 + 6 + 8 \;=\; 3 + 13 \;=\; 4 + 12.$$

However, this sequence is not 3-composite, as the sums $1 + 4 + 13 + 8$, $\;2 + (-2) + 12$, and $3 + 6 + 17$ do not give equal outcomes.

(a) Give a sequence of 6 *distinct* integers that is both 2-composite and 3-composite.

(b) Give a sequence of 7 *distinct* integers that is 2-composite, 3-composite, and 4-composite.

(c) Find the largest $k \leqslant 99$ for which there exists a sequence of 99 *distinct* integers that is k-composite. (Give an example of such a sequence and prove that such a sequence does not exist for greater values of k.)

C2. A year is called *interesting* if it consists of four distinct digits. For example, the year 2019 is interesting. It is even true that all years from 2013 up to and including 2019 are interesting: a sequence of seven consecutive interesting years.

(a) Determine the next sequence of seven consecutive interesting years and prove that this is indeed the next such sequence.

(b) Prove that there is no sequence of eight consecutive interesting years within the years from 1000 to 9999.