

# Second round

## Dutch Mathematical Olympiad



Friday 11 March 2016

### Solutions

#### B-problems

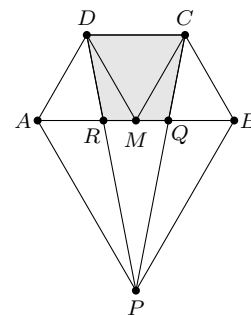
**B1.** 4 The first digit of the integer is called  $a$  and the second digit is called  $b$ . The condition then becomes  $10 \cdot a + b = 4(a + b)$ . Rewriting this we get  $6a = 3b$ , or  $2a = b$ . Because  $0 \leq b \leq 9$ , the only possibilities are  $a = 0, 1, 2, 3, 4$ . However,  $a = 0$  cannot hold because we are looking for integers between 10 and 99. Hence, we find the following four solutions: 12, 24, 36 and 48.

**B2.** 68 Of the 100 cards there are 33 with a number that is divisible by 3 and 50 cards with a number that is divisible by 2. In other words: there are exactly  $100 - 33 = 67$  cards with a number that is *not* divisible by 3 and  $100 - 50 = 50$  cards with a number that is *not* divisible by 2. If Lisa picks 68 cards, then at least one of the chosen cards has a number that is divisible by 3, because  $68 > 67$ . As  $68 > 50$ , also one of the chosen cards must contain a number that is divisible by 2. This implies that the product of the 68 numbers on the chosen cards must be divisible by 6.

On the other hand, if Lisa takes 67 or fewer cards, then it could happen that all of these cards have a number that is not divisible by 3. But then the product of these numbers is not divisible by 3, and hence also not divisible by 6.

We conclude that Lisa has to take at least 68 cards from the box to be absolutely sure that the product of the numbers on these chosen cards is divisible by 6.

**B3.** 5 Let  $M$  be the midpoint of  $AB$ . Because  $|AM| = |CD|$  and  $AM$  and  $CD$  are parallel, the quadrilateral  $AMCD$  is a parallelogram. It follows that  $|CM| = |AD|$ , hence triangle  $BCM$  is equilateral. Hence, the area of triangle  $BMC$  is a quarter of the area of triangle  $ABP$  and hence it equals 3.



Triangles  $MQC$  and  $BQP$  are similar because angles  $CMQ$  and  $PBQ$  are both 60 degrees and  $MQC$  and  $BQP$  are equal opposite angles (AA). From  $|BP| = 2|CM|$  it follows that  $|BQ| = 2|MQ|$ , which implies that  $|MQ| = \frac{1}{3}|MB|$ . Hence, the area of triangle  $CMQ$  is  $\frac{1}{3}$  of the area of triangle  $BMC$  and therefore it equals  $\frac{1}{3} \cdot 3 = 1$ . Using a similar reasoning, it follows that also triangle  $DRM$  has an area of 1. The area of triangle  $CDM$  is 3 because it has the same area as triangle  $MBC$  (equal lengths of bases and heights).

The area of quadrilateral  $QCDR$  is the sum of the areas of triangles  $RMD$ ,  $MQC$ , and  $MCD$ . We conclude that the area of  $QCDR$  is  $1 + 1 + 3 = 5$ .

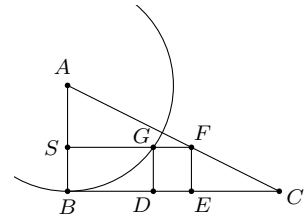
- B4.** 7 The contestants in the championship were three girls and a number of boys. Call one of these boys Julian. In each game, Julian played against exactly one girl. Since he didn't play against the same girl twice, he played at most 3 games. In total, he therefore played against at most 3 other boys (one per game). Since he also played against each other boy, there can be at most 3 other boys. Hence, the total number of contestants is at most  $1 + 3 + 3 = 7$ .

A situation with 7 contestants is indeed possible. If we call the girls  $A, B,$  and  $C,$  and the boys  $J, K, L,$  and  $M,$  then the following seven games constitute a championship:

$$ABC, \quad AJK, \quad ALM, \quad BJL, \quad BKM, \quad CJM, \quad CKL.$$

It is straightforward to check that in this setup each pair of contestants play against each other in exactly one game.

- B5.**  $\frac{2}{5}$  Let  $x$  be the length of the side of the square. Triangle  $FEC$  is similar to  $ABC$ . This yields  $|EC| = 2|EF| = 2x$ . We now see that  $|BD| = |BC| - |DE| - |EC| = 2 - 3x$ . We extend segment  $FG$  of the square to a line and let  $S$  be the intersection with  $AB$ . We see that  $|AS| = |AB| - |BS| = 1 - |EF| = 1 - x$ . Observe that angle  $ASG$  is a right angle. Because of the Pythagorean theorem we have  $|AG|^2 = |SG|^2 + |AS|^2 = |BD|^2 + (1 - x)^2$ , or  $1^2 = (2 - 3x)^2 + (1 - x)^2$ , because  $|AG| = |AB| = 1$ . Expanding this yields  $10x^2 - 14x + 4 = 0$ . The two solutions are  $x = 1$  and  $x = \frac{2}{5}$ . Because  $x < |AB| = 1$  we have that  $x = \frac{2}{5}$  is the only solution.



## C-problems

- C1.** When adding two integers, we write one on top of the other and then add the digits from right to left. If at a certain position the sum of the two digits is greater than 9, a *carry* occurs at this position: we must *carry* a 1 to the addition of the two digits in the next position (the position to the left).

Let  $n$  be a number whose sum of digits equals  $s$ . Now consider the number  $n + 2016$ . We can derive the sum of its digits from the addition procedure. If no carry occurs, then the sum of the digits of  $n + 2016$  equals  $s + 2 + 1 + 6 = s + 9$ . If carries do occur, then with each carry the sum of the digits decreases by 9. After all, if two digits  $x$  and  $y$  add up to a number greater than 9, the digit below these two will be  $x + y - 10$  instead of  $x + y$ , while we carry a 1 causing the digit to the left to increase by 1. In total, the sum of the digits thus decreases by 9 for each carry.

For example, taking  $n = 1015$  (and  $s = 1 + 0 + 1 + 5 = 7$ ), then there will be one carry in the addition  $n + 2016 = 3031$ . For the sum of the digits we have  $3 + 0 + 3 + 1 = (s + 9) - 1 \cdot 9$ .

As another example, take  $n = 3084$  (where  $s = 15$ ). Then, there will be two carries in the addition  $n + 2016 = 5100$  and indeed we have  $5 + 1 + 0 + 0 = (s + 9) - 2 \cdot 9$ .

Hence, a number is 2016-invariant if and only if exactly one carry occurs when adding 2016 to it.

- (a) After all observations in the previous paragraph, we conclude that we are looking for the greatest four-digit number having the property that you need to carry exactly one 1 when adding 2016 to it. This number is 9983. In the addition  $9983 + 2016$  exactly one carry occurs and if you raise one of the digits 8 or 3, a second carry occurs.

(b) To count how many of the numbers  $n$  from 1 to 9999 give exactly one carry when adding 2016 to it, we consider four cases according to the position in which the carry occurs. In each case we consider the possible values of the digits of  $n$ .

- First, consider the case that the only carry occurs in the thousands place. The thousands digit must then be either 8 or 9. The unit digit can only be 0, 1, 2, or 3. The tens digit can be any digit except for 9 (because  $1 + 9$  would give a carry in the tens place). The hundreds digit can be any value. Hence, in total there are  $2 \cdot 10 \cdot 9 \cdot 4 = 720$  numbers in this case.
- There are no numbers for which a carry occurs only in the hundreds place (a carry there can only occur if we also carry a 1 from the tens place).
- Next, consider the case that the only carry occurs in the tens place. For the units digit, the only possible values are 0, 1, 2, and 3. The tens digit must be a 9. Because you now need to carry a 1 from the tens place to the hundreds place, the digit in the hundreds place cannot be a 9 anymore since this would cause another carry (for example, in the addition  $990 + 2016$ ). The digit in the thousands place can take all values except 8 and 9. Hence, in total there are  $8 \cdot 9 \cdot 1 \cdot 4 = 288$  numbers in this case.
- Finally, consider the case that the only carry occurs in the units place. For this digit, the possible values are 4, 5, 6, 7, 8, and 9. The digit in the tens place cannot be 9 or 8, because then another carry would occur (for example in  $84 + 2016$ ). The hundreds digit can be any value and the thousands digit can be any value except 8 and 9. Hence, in total we find  $8 \cdot 10 \cdot 8 \cdot 6 = 3840$  numbers in this case.

Altogether, there are  $720 + 288 + 3840 = 4848$  numbers from 1 to 9999 that are 2016-invariant.

**C2.** There are four students seated in the corners, since  $n, m \geq 2$ . They each shook the hand of three other people. Not counting the students in the corners, there are  $m - 2$  students who are seated completely in the front,  $m - 2$  who are seated in the back,  $n - 2$  students who are seated in the leftmost row and  $n - 2$  in the rightmost row. These  $2n + 2m - 8$  students each shook the hand of five other people. The remaining  $mn - 4 - (2n + 2m - 8) = mn - 2n - 2m + 4$  students each shook the hand of eight other people. Adding the numbers of handshakes for each of the students gives a total of

$$4 \cdot 3 + (2n + 2m - 8) \cdot 5 + (mn - 2n - 2m + 4) \cdot 8 = 8mn - 6n - 6m + 4.$$

This number is exactly twice the number of handshakes, because each handshake involves two students. Hence, we see that  $8mn - 6n - 6m + 4 = 2040$ , or  $16mn - 12n - 12m = 4072$ . We can rewrite this equation as

$$(4m - 3) \cdot (4n - 3) = 4081.$$

The integers  $4m - 3$  and  $4n - 3$  both have remainder 1 when dividing by 4. Also,  $4m - 3$  and  $4n - 3$  must both be greater than 1. If we look at the prime factorisation  $4081 = 7 \cdot 11 \cdot 53$ , then we see that there is only one way to write 4081 as the product of two integers greater than 1 that each have remainder 1 when dividing by 4: namely as  $4081 = 77 \cdot 53$ . Hence, we find  $4m - 3 = 77$  and  $4n - 3 = 53$  (or the other way around). It follows that  $m = 20$  and  $n = 14$  (or the other way around). In both cases we see that the total number of students equals  $20 \cdot 14 = 280$ .