Nederlandse Wiskunde Olympiade voor Bedrijven

Friday, 31 January 2025

Solution uitsmijter

Problem

Determine all (not necessarily positive) integers n such that $14n^2 + 37n + 46$ is divisible by $5n + 12$.

Answer

$n = 2$, $n = -2$, $n = -11$ and $n = -97$.

Solution

Suppose that $14n^2 + 37n + 46$ is divisible by $5n + 12$, i.e. $5n + 12$ is a divisor of $14n^2 + 37n + 46$. Then $5n + 12$ is also a divisor of

$$5 \cdot (14n^2 + 37n + 46) = 70n^2 + 185n + 230.$$

Now we subtract a suitable multiple of $5n + 12$ such that the term $70n^2$ disappears. This does not change divisibility by $5n + 12$. Therefore $5n + 12$ is also a divisor of

$$(70n^2 + 185n + 230) - 14n \cdot (5n + 12) = (70n^2 + 185n + 230) - (70n^2 + 168n) = 17n + 230.$$

Multiplying by 5 shows that $5n + 12$ is also a divisor of $5 \cdot (17n + 230) = 85n + 1150$. It follows that $5n + 12$ is a divisor of

$$(85n + 1150) - 17 \cdot (5n + 12) = (85n + 1150) - (85n + 204) = 946.$$

The divisors of $946 = 2 \cdot 11 \cdot 43$ are -946, -473, -86, -43, -22, -11, -2, -1, 1, 2, 11, 22, 43, 86, 473 and 946. Therefore $5n + 12$ must be equal to one of these divisors. This is only possible if that divisor has remainder 2 when divided by 5. It follows that the divisor must be -473, -43, 2 or 22 and hence the possible solutions for n are $n = -97$, $n = -11$, $n = -2$ and $n = 2$.

Now we have to show that these n satisfy the condition. There are two ways to do this. The first way is to fill in the possible values for n and check the divisibility relation. We find:

n	$5n + 12$	$14n^2 + 37n + 46$	divisible?
2	22	176	yes
-2	2	28	yes
-11	-43	1333	yes
-97	-473	128,183	yes

So these n are indeed solutions.
Remark: note that the computations can be simplified by using that $14n^2 + 37n + 46$ is divisible by $5n + 12$ if and only if

$$(14n^2 + 37n + 46) - 3n \cdot (5n + 12) = (14n^2 + 37n + 46) - (15n^2 + 36n) = -n^2 + n + 46$$

is divisible by $5n + 12$.

It is also possible to show that these n satisfy the condition without computing $14n^2+37n+46$. The first part of the solution consists of four steps: two steps in which $14n^2+37n+46$ and $17n+230$ are multiplied by 5, and two steps in which a multiple of $5n+12$ is subtracted. It is clear that subtracting a multiple of $5n+12$ does not change divisibility by $5n+12$, so these two steps do not introduce 'additional' possibilities for n. The same holds for the other two steps: since $5n+12$ and 5 have no common non-trivial divisors, $5n+12$ can only by a divisor of $5 \cdot (14n^2+37n+46)$ or $5 \cdot (17n+230)$ if it is a divisor of $14n^2+37n+46$ or $17n+230$, respectively. Therefore all possible solutions n indeed satisfy the condition.

We conclude that there are four solutions: $n=-97$, $n=-11$, $n=-2$ en $n=2$.