First round
Dutch Mathematical Olympiad

16–27 January 2023

Solutions

A1. B) 4 No two of the four X's in the corners can be connected to the same mast. So at least four masts are needed. On the right you can see a solution with four masts.

X		X		X
	M		M	
X				X
	M		M	
X		C		X

A2. D) 4 Since three of the four items have a price in whole euros, either 6 of the sticker sheets have been sold (for 1.80 euros) or 16 (for 4.80 euros). More is not possible, because there were only 20 in stock. So for the other three items, there are either 74 or 71 euros left. First assume there are 74 euros left. Since the teddy bears and water guns cost a multiple of 5 euros, 3 of the footballs must have been sold (65 euros left) or a multiple of 5 more. But 8 footballs is impossible, because then more than 8 teddy bears must have been sold and there were not that many in stock. So 3 footballs have been sold. Then at least 4 water guns must have been sold and at least 4 teddy bears, but that is more than 65 euros together. So this is not possible.

Now assume there are 71 euros left. Then, by the same reasoning, 2 footballs must have been sold or at least 7, but the latter is again impossible. So there are 65 euros left for the teddy bears and water gunss. Of these, at least 3 must have been sold, for at least $3 \cdot 5 + 3 \cdot 15 = 60$ euros. So exactly 3 water gunss must have been sold, and to complete the amount, exactly 4 teddy bears.

A3. C) 4 On each square of the chessboard, we write down how many moves it takes a knight to get there. We start with 0 in the corner, then write down a 1 on the (empty) squares where you can get from the 0 in one jump, then write down a 2 on the (empty) squares where you can get from a 1 in one jump, and so on. In the end, we count how many times each number occurs and see that 4 is the most occurring number, namely 21 times (3 occurs 20 times).

5	4	5	4	5	4	5	6
4	3	4	3	4	5	4	5
3	4	3	4	3	4	5	4
2	3	2	3	4	3	4	5
3	2	3	2	3	4	3	4
2	1	4	3	2	3	4	5
3	4	1	2	3	4	3	4
0	3	2	3	2	3	4	5

A4. E) 3 Denote the three equal angles by α. We see that $|EB| = |ED| + |DB| = 2 + 4 = 6 = |AB|$, and so triangle $\triangle ABE$ is an isosceles triangle. It follows that $\angle AEB = \angle BAE = 2\alpha$ and because of the straight angle at E, we find that $\angle AEC = 180° - 2\alpha$. Since $\angle CAE = \alpha$ and the sum of the angles in $\triangle ACE$ equals 180°, it also holds that $\angle ACE = \alpha$. So we have found another isosceles triangle, namely $\triangle ACE$, and it follows that $|CE| = |AE| = 3$.

A5. C) 220 We can make letters A at 10 different heights. For each height, we keep track of how many there are. For height 1, there are 10 places where we can put the legs of the A; the horizontal dash can then only go in one place. For height 2, there are 9 places where the legs can go; the horizontal dash can now go in 2 possible places. For height 3, there are 8 places where the legs can go; the horizontal dash can go in 3 possible places. This pattern continues and in total we find $1 \cdot 10 + 2 \cdot 9 + 3 \cdot 8 + 4 \cdot 7 + 5 \cdot 6 + 6 \cdot 5 + 7 \cdot 4 + 8 \cdot 3 + 9 \cdot 2 + 10 \cdot 1 = 220$ possible letters A.

A6. B) 506 The first person cannot lie, because then they would be a knave and the first statement would be true. So this person is a knight. Therefore, because of this person's second statement, there are at least 1012 knights. All 1011 other people who make a statement about the number of knights are therefore speaking the truth and are themselves knights.

So all knaves are among the 1012 people who make a statement about the number of knaves. Suppose there are exactly k knaves. Then, of those 1012 people, exactly k speak the truth and the other $1012 - k$ lie. So the latter are the knaves, so $1012 - k = k$ and it follows that $k = 506$.

A7. C) 3 Since $a < b$, we have $\frac{1}{a} > \frac{1}{b}$. In particular, $\frac{1}{a}$ must therefore be more than half of $\frac{4}{15}$, i.e., $\frac{1}{a} > \frac{2}{15}$. It follows that $a < \frac{15}{2} < 8$. Moreover, a must be at least 4, otherwise $\frac{1}{a}$ is greater than $\frac{4}{15}$. Hence we have four possibilities for a: 4, 5, 6 and 7. For $a = 4$, we find $\frac{1}{b} = \frac{4}{15} - \frac{1}{4} = \frac{1}{60}$ and thus $b = 60$. For $a = 5$, we find $\frac{1}{b} = \frac{4}{15} - \frac{1}{5} = \frac{1}{15}$ and thus $b = 15$. For $a = 6$, we find $\frac{1}{b} = \frac{4}{15} - \frac{1}{6} = \frac{1}{10}$ and thus $b = 10$. For $a = 7$, we find $\frac{1}{b} = \frac{4}{15} - \frac{1}{7} = \frac{13}{105}$ and in this case the solution $b = \frac{105}{13}$ is not an integer. So in total we find 3 solutions.

A8. B) 540 Fix an order for the pencils. We can write a distribution of the pencils among the girls as a sequence of six times the letter A, B, or C. For example, AABBCA means Anna gets the first, second and sixth pencil, Bella gets the third and fourth pencil, and Celine gets the fifth pencil. There are $3^6 = 729$ of these sequences, but not all of these sequences give an allowed distribution, because every girl needs to get at least one pencil.

There are three sequences where one girl gets all the pencils: AAAAAA, BBBBBB and CCCCCC. Those we do not want to count. Next we consider how many sequences there are where Celine does not get a pencil, but Anna and Bella both get at least one pencil. There are 2^6 possibilities for a sequence with only A and B, but among those we have already seen AAAAAA and BBBBBB. So there are $2^6 - 2 = 62$ sequences where Anna and Bella both get at least one pencil, but Celine gets none. Similarly we find that there are 62 sequences where Bella or Anna does not get a pencil, but the other two girls both get al least one pencil.

In total we find there are $729 - 3 - 3 \cdot 62 = 540$ different ways to distribute the pencils.

B1. 1 Note that the median of 2023 numbers is always the number in spot 1012 if you sort the numbers by size. The first number that Albert adds, which therefore comes in place 2024, is the median of the numbers 1 through 2023. That is 1012. We will now show that all subsequent numbers Albert adds are also equal to 1012.

Suppose Albert has just written the number 1012 at spot n, as the median of the numbers at spot $n - 2023$ to $n - 1$. We call this the old set. Which number will be in place $n + 1$? To do this, Albert looks at the set of numbers in place $n - 2022$ to n, which we call the new set. The new set is almost identical to the old set: the number 1012 has been added and the number in place $n - 2023$ is no longer there. Call the latter number x.

We now consider the three options for x. If $x = 1012$ then the new set consists of exactly the same numbers as the old set, and so the median is 1012 again. If x is greater than 1012, then the smallest 1012 numbers from the old set remain the same (since the median of the old set is 1012), and so the median is 1012 again. If x is smaller than 1012, then the largest 1012 numbers in the new set are exactly the largest 1012 numbers from the old set, and again the median equals 1012.

We conclude that Albert writes down the number 1012 every time. Therefore, even the three thousandth number equals 1012. Thus, there is only one possibility.

B2. 175 We assume that the three-digit number consists of the digits a, b, and c and is therefore equal to $100a + 10b + c$. The digit product of this number is $a \cdot b \cdot c$. So it must hold that $100a + 10b + c = 5 \cdot a \cdot b \cdot c$. Since the right hand side of this equation is a multiple of five, it must hold that $c = 0$ or $c = 5$. If $c = 0$ then also $5 \cdot a \cdot b \cdot c = 0$ and that cannot be the case, because we started with a number unequal to zero. So $c = 5$. That means that $5 \cdot a \cdot b \cdot c$ is a multiple of 25 that ends in a 5, so it ends in 25 or 75, so $b = 2$ or $b = 7$. Moreover, from the fact that $100a + 10b + c = 5abc$ ends in a 5, we find that a, b and c are all odd, so $b = 7$. Filling in the values found for b and c into the first equation, we find $100a + 75 = 175a$. The only solution to this equation is $a = 1$. So the only three-digit number that is exactly five times its digit product is 175.

B3. $\frac{1}{2}$ If we rotate the second-largest square 45 degrees, the vertices lie on the centres of the sides of the largest square, and we see that the largest square has twice the area of the second-largest square. This situation repeats itself: the second largest square in turn has twice the area of the third largest square.

If the area of the third square is A, then the area of the second square is $2A$ and that of the first square is $4A$. So the outer light grey area is $2A - A = A$ and the outer dark grey area is $4A - 2A = 2A$. So the outer dark grey area is twice the size of the outer light grey area. The next dark grey area is twice the size of the next light grey area, and so on. So in total, the dark grey area has twice the size of the light grey area. So the light grey area has area $\frac{1}{2}$.

B4. 1113 Since both a^2 and b^2 have only seven digits, the digits of a and b cannot be too large. After all, $4000^2 = 16{,}000{,}000$, so the first digit of both a and b is 1, 2, or 3. Since you get b by reading a backwards, the last digit of both a and b is also equal to 1, 2 or 3. If the first and last digits of a are the same, then so are the first and last digits of b, and moreover equal to the first and last digits of a. But then the last digit of a^2 is equal to the last digit of b^2, and that cannot be true since a^2 and b^2 each have seven different digits, in reverse order. So the first and last digit of a are not equal. This means that both a and b can only contain the digits 1, 2, and 3. The remaining possibilities for a are now 1112, 1113, 1222, 1333, 2223, and 2333. The last three possibilities are immediately ruled out because the square of b would be at least 3200^2 and that already has eight digits. Trying out the first three possibilities gives $a = 1113$ as the only possibility. In that case $a^2 = 1{,}238{,}769$, $b = 3111$, and $b^2 = 9{,}678{,}321$.