First round
Dutch Mathematical Olympiad

18 January – 4 February 2021

Solutions

A1. D) 7 From the information in the problem, we deduce that $a-b=\pm 2$, $b-c=\pm 3$, and $c-d=\pm 4$. This means that

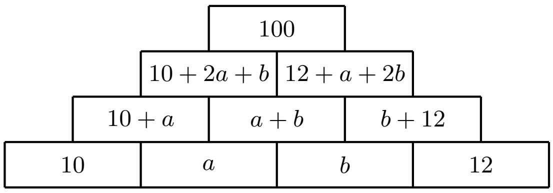
$$a-d=(a-b)+(b-c)+(c-d)=\pm 2\pm 3\pm 4.$$

We can try all possibilities for the plus and minus signs:

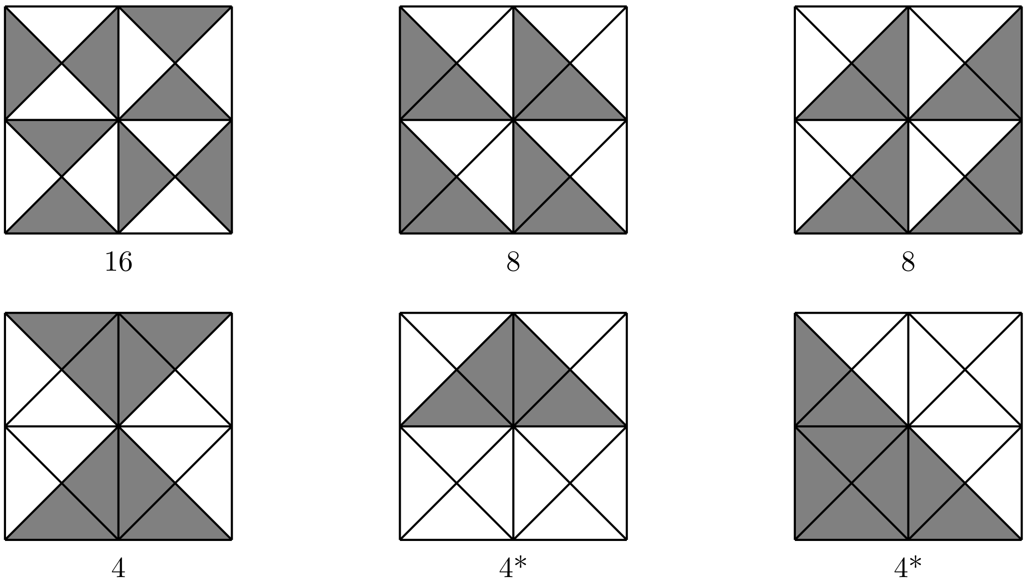
$$2+3+4=9,\quad 2+3-4=1,\quad 2-3+4=3,\quad 2-3-4=-5,$$
$$-2-3-4=-9,\quad -2-3+4=-1,\quad -2+3-4=-3,\quad -2+3+4=5.$$

The only value that does not occur (neither with a plus sign nor with a minus sign) is 7.

A2. D) 26 The two missing numbers in the bottom row are called a and b. We fill the rest of the pyramid, like in the figure on the right. On the top is the number $10+2a+b+12+a+2b=22+3(a+b)$. Because we know that $a+b=x$ and the top number must equal 100, we find that $100=22+3x$, or $x=26$.



A3. D) 44 The triangles appear in four different sizes. From small to large the numbers of triangles are 16, 16, 8 and 4. This is illustrated in the figures below. The total number of triangles is 44.



* = By rotating this figure in four different ways, you get four different triangles.

A4. B) 2 Note that the building of height 5 is always visible. First we consider the middle column, which has a 6 below it. We must have that $6 = 1 + 5$: the first building that is visible has height 1, and behind it you see a building of height 5. There can be no other building in between them, because such a building would also be visible from the side. In this way, we obtain the numbers in the first figure.

Now we consider the second row, that has an 8 next to it at the right. The building of height 1 in this row can only be in the second or fifth column, because of the first requirement. If the 1 is in the fifth column, the 8 on the right is only correct if the order of the buildings is 4 – 3 – 5 – 2 – 1. This is impossible, because there already is a 5 in the third column. Hence, the 1 is in the second column. Now we can only obtain the sum 8 on the right side if the buildings of height 3 and 5 are visible. Hence, the 3 is in the fifth column and because of the first requirement, the 5 is in the fourth column. Now we got the numbers in the second figure.

	8				
4					**8**
			1		
		5		**?**	
3		**1**			
		6			

	8				
4	**1**	**2**	**5**	**3**	**8**
			1		
		5		**?**	
3		**1**			
		6			

	8				
	3				
4	**1**	**2**	**5**	**3**	**8**
	5		**1**		
		5		**?**	
3		**1**			
		6			

Finally we consider the 8 above the second column. There is already a 1 in the second row. This building cannot be visible. Hence, the visible buildings in this column are the buildings of height 3 and 5. The building of height 3 must be in the top row, and the 5 in the third row, because the fourth row already contains a 5 and the fifth row is not allowed for the 5, as the building of height 4 would also be visible in that case.

We have now filled the numbers as in the third figure. Continue putting the numbers like you solve a sudoku: each number must occur exactly once in each row and each column. The first column, the third column and the middle row can be filled completely. On the space with the question mark must be the number 2.

A5. B) 15 We will reason backwards, starting with the last turn. The number 2021 is odd, so the last number that was written down before it has to be 2022. For the number before that there are two possibilities: 2023 and 1011. We will show that, if the solution has the minimum number of turns, this number has to be 1011. In other words: if we can divide by two in our backwards reasoning, this is always the best turn.

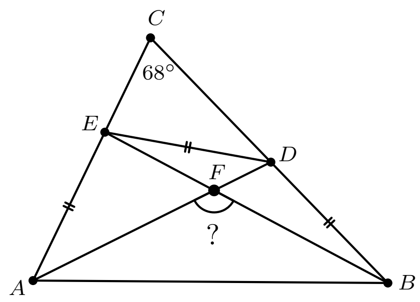
Suppose we add 1 to 2022 a number of times before we divide by 2: say we add n times a 1 before dividing by 2. This leaves the number $\frac{2022+n}{2}$. This has to be an integer, so n needs to be divisible by 2. The number $\frac{2022+n}{2}$ is also equal to $1011 + \frac{n}{2}$. So instead of adding n times a 1 and then dividing by 2, we could have also divided by 2 and added $\frac{n}{2}$ times a 1. Because $\frac{n}{2}$ is less then n, this takes less turns. So it is always optimal to first divide by 2.

The numbers we get from our backwards reasoning are:

$$2021, 2022, 1011, 1012, 506, 253, 254, 127, 128, 64, 32, 16, 8, 4, 2, 1.$$

We see we need at least 15 turns to write down 2021.

A6. E) 124° First we consider the left side of the picture. We see that $\triangle AED$ is an isosceles triangle, and hence $\angle EAD$ and $\angle EDA$ have the same size. The sum of the angles in a triangle is 180°, hence $\angle AED + 2 \cdot \angle EDA = 180°$. Because $\angle AEC$ is a straight angle, we also have $\angle AED + \angle CED = 180°$. This yields $\angle CED = 2 \cdot \angle EDA$.



Using the same reasoning for the right side of the picture, starting from the isosceles triangle $\triangle BDE$, we find that $\angle CDE = 2 \cdot \angle DEB$. Because the sum of the angles in the top triangle $\triangle ECD$ is 180°, we get that

$$\begin{aligned} 180° &= \angle ECD + \angle CED + \angle CDE \\ 180° &= 68° + 2 \cdot \angle EDA + 2 \cdot \angle DEB \\ 112° &= 2 \cdot (\angle EDA + \angle DEB) \\ 56° &= \angle EDA + \angle DEB \end{aligned}$$

Now consider triangle $\triangle EFD$. The sum of the angles is 180°, and two of the angles are already part of other triangles. We obtain

$$\begin{aligned} 180° &= \angle DEF + \angle EDF + \angle EFD \\ 180° &= \angle DEB + \angle EDA + \angle EFD \\ 180° &= 56° + \angle EFD \\ 124° &= \angle EFD \end{aligned}$$

Finally, we note that $\angle AFB$ and $\angle EFD$ are opposite angles, so they must have the same size. Hence, the angle at the question mark is 124°.

A7. A) 6 The average of the numbers 1 to n is exactly $\frac{n+1}{2}$. This is because we can pair up the numbers, with one number left out in the middle if n is odd: 1 pairs with n, 2 pairs with $n-1$, etcetera. The average of each pair is $\frac{n+1}{2}$.

If we wipe out a number, the smallest possible average of the remaining numbers is $\frac{n}{2}$. This is the average of the numbers 1 to $n-1$. In this case, we have wiped out the largest number. The largest possible average occurs when we wipe out 1. The average of the numbers 2 to n is $\frac{n+2}{2}$.

We see that $\frac{n}{2} \leqslant \frac{45}{4}$ and $\frac{45}{4} \leqslant \frac{n+2}{2}$. Multiplying both inequalities by 2 we get $n \leqslant \frac{45}{2} < 23$ and $22 < \frac{45}{2} \leqslant n+2$, so $n > 20$. This leaves two possibilities: $n = 21$ and $n = 22$. We will show that $n = 22$ is impossible.

Suppose $n = 22$. Then there would be 21 numbers left. In taking the average of 21 integers, we have to add 21 integers and divide the result by 21. This gives a fraction of the form $\frac{S}{21}$. That can never be equal to $\frac{45}{4}$, for the equation $\frac{S}{21} = \frac{45}{4}$ gives $4S = 21 \cdot 45$, but $21 \cdot 45$ is not divisible by 4.

This means we started with the numbers 1 to 21 written on the board. The sum of them is exactly 21 times their average: $21 \cdot \frac{21+1}{2} = 21 \cdot 11 = 231$. The average of the remaining numbers has to be $\frac{45}{4}$, so the sum of the remaining numbers needs to be $20 \cdot \frac{45}{4} = 225$. This means we have to wipe out the number $231 - 225 = 6$.

A8. D) 2027 The numbers on the diagonal starting from the top left corner are exactly the squares of the odd numbers (1, 9, 25, 49, etcetera). Moreover, if a certain odd number is on the diagonal, then the next odd number is one square to the right and one square above it.

The number in row n and column n is $(2n-1)^2$. In row 23 and column 23, there is the number $45^2 = 2025$. The number in row 22 and column 24 is one square to the right and one square above it. Hence, that number is the next odd number: 2027.

B1. {14, 19}, {15, 18} and {16, 17} We call the digits of our two numbers a, b, c, and d, so the first number is equal to $10a + b$ and the second number is equal to $10c + d$. We have

$$S = 10(a + c) + b + d.$$

The numbers we get by interchanging the digits are $10b + a$ and $10d + c$. Adding them gives

$$4S = 10(b + d) + a + c.$$

Multiplying the first equation by -4 and adding it to the second equation, gives

$$0 = -39(a + c) + 6(b + d), \qquad \text{and thus} \qquad 13(a + c) = 2(b + d).$$

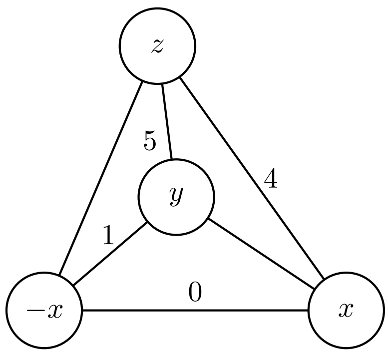
The right-hand side of the last equation can only be divisible by 13 if $b + d$ is divisible by 13. Since $b + d$ is at least 0 and at most 18, this can only happen if $b + d = 0$ or $b + d = 13$.

If $b + d = 0$, then we have $a + c = 0$. In that case, both numbers start with a 0, but that was not allowed. So this is not possible.

If $b + d = 13$, then we get $a + c = 2$. The digits a and c cannot be zero. Hence we have $a = c = 1$. The possibilities for the pair $\{b, d\}$ are $\{4, 9\}$, $\{5, 8\}$ and $\{6, 7\}$. So the possibilities for the pair of numbers at the start are $\{14, 19\}$, $\{15, 18\}$ and $\{16, 17\}$. It is not difficult to check that all of these are in fact solutions. In all cases we get $S = 33$ if we add the two numbers, and $132 = 4S$ if we add the numbers with interchanged digits.

B2. -6 and $-\frac{21}{16}$ If we add the six numbers on the line segments, then we have counted each of the numbers in the circles three times. Because $0 + 1 + 2 + 3 + 4 + 5 = 15$, the sum of the four numbers in the circles is 5. Moreover, if we consider two line segments that do not meet each other in a circle, so they touch each of the four circles once, their sum must equal the sum of the four numbers in the circles, which is 5. Hence, for the three pairs of non-adjacent line segments, the possible combinations are 0-5, 1-4, and 2-3.

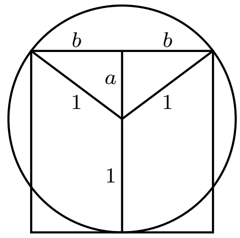
The numbers in the circles that are adjacent to the line segment 0 differ by a factor of -1. We denote them by x and $-x$. The other two numbers in the circles are called y and z. So we have to attach the 5 to the line segment between y and z. Because of symmetry, it does not matter if we swap x and $-x$, or y and z. Hence, without loss of generality, we can place the pair 1-4 as in the figure on the right.



Now there are only two ways to assign the 2 and the 3 to the line segments. If we assign a 3 to the line segment between x and y, then we get $y - x = 1$ and $y + x = 3$. When we add these two equations, we obtain $2y = 4$, or $y = 2$. Then we also get $x = 1$, $-x = -1$ and $z = 3$. The product of the four numbers in the squares is $-1 \cdot 1 \cdot 2 \cdot 3 = -6$ in this case.

To get the second solution, we assign the 2 to the line segment between x and y. This gives $y - x = 1$ and $y + x = 2$. When we add these two equations, we get $2y = 3$, or $y = \frac{3}{2}$. Then it follows that $x = \frac{1}{2}$, $-x = -\frac{1}{2}$, and $z = \frac{7}{2}$. The product of the four numbers in the circles is $-\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{7}{2} = -\frac{21}{16}$ in this case.

B3. $\frac{8}{5}$ In the figure on the right, we gave names to the different lengths. The length from the centre of the circle to the upper side of the square is a and half of the side of the square is b. All segments of length 1 are the radius of the circle.



Using the Pythagorean theorem we find that $a^2 + b^2 = 1$. Moreover, $a + 1$ is exactly the length of the side of the square, so $a + 1 = 2b$ and $a = 2b - 1$. Substituting this in the first equation gives $(2b - 1)^2 + b^2 = 1$. This can be simplified to $5b^2 - 4b = 0$.

Because $b \neq 0$, we can divide the equation by b to get $5b - 4 = 0$, therefore $b = \frac{4}{5}$. The length of the side of the square is thus $2 \cdot \frac{4}{5} = \frac{8}{5}$.

B4. 37 To get an impression of the number of colours we need, we will construct a sequence of codes in which each code dominates all previous codes. We start with 0000 and end with 9999. For example, you could find the sequence

$$0000, 0001, 0002, \ldots, 0009, 0019, \ldots, 0099, 0199, \ldots, 0999, 1999, \ldots, 9999.$$

There are many other possible sequences from 0000 to 9999. The codes in this sequence must get distinct colours. Hence, for this sequence, we need at least 37 colours.

Now we will show that 37 colours are sufficient to colour all possible security codes. If we add the four digits of a code, we get a number which we will call the *power* of the code. For example, code 0000 has power 0 and 9999 has power 36. All codes have a power between 0 and 36. If code A is dominated by another code B, then the power of B is greater than the power of A. *(The converse is false: if code B has greater power than code A, then code B does not have to dominate code A. For example, consider code* 2021 *with power* 5*, and code* 0089 *with power* 17*:* 2021 *is not dominated by* 0089*.)*

We will use the power to colour the codes. For each value of the power, we use a different colour. That is: all codes with the same power get the same colour, and codes with different powers get different colours. In this way, it can never happen that code A is dominated by another code, say B, when A and B have the same colour. The power of a code is a number from $0, 1, \ldots, 36$. Hence, using 37 colours, we can colour all codes. We already saw that we need at least 37, hence this is also the minimum number of colours that is needed.