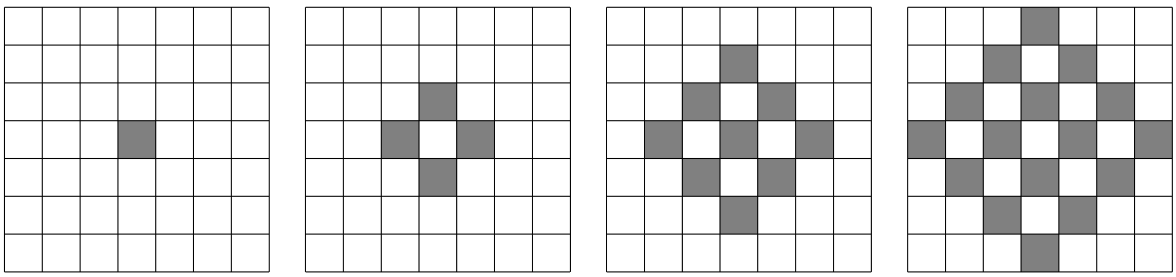
First Round
Dutch Mathematical Olympiad

21 January – 31 January 2019

Solutions

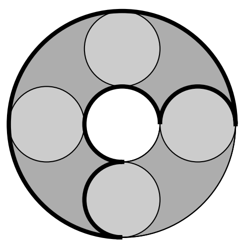
A1. E) 5 Of the pair of numbers 1 and 9, Arthur can have written only one, because no two of the written numbers may add up to 10. The same is true for the pair 2 and 8, for 3 and 7, and for 4 and 6. It follows that Arthur can have written down only four of the eight numbers 1, 2, 3, 4, 6, 7, 8, and 9. This means that he must have written down 5, the only remaining number.

A2. E) 10000 Looking at the squares which are sick on day 1, 2, 3, and 4, we see that these squares form a square rotated over $45°$ (see below). On day 1, there is $1^2 = 1$ sick square. On days 2, 3, and 4, there are $2^2 = 4$, $3^2 = 9$, and $4^2 = 16$ sick squares, respectively. On day 5, the same squares are sick as on day 3, together with an extra layer of new sick squares, turning it into a 5×5-square. The pattern continues this way until on day 100 there are $100^2 = 10000$ sick squares.



day 1 day 2 day 3 day 4

A3. D) 5π In the right figure, we see that the big circle consists of five smaller circles and four dark grey pieces. The big circle has area 9π and each of the smaller circles has area π, hence the four dark grey pieces together have area $9\pi - 5\pi = 4\pi$. Since all four grey pieces have the same area, each grey piece has area π. We conclude that the total area of the figure in the problem is 5π (two small circles and three dark pieces together).



A4. B) 7 The thirteen multiples are: 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, and 98.

The multiples 70 and 77 only consist of the digits 0 and 7. Moreover, these are the only multiples with a 0 or 7. Hence, a chain with either 70 or 77 can have length at most 2.

Analogously, the multiples 35, 56, and 63 are the only multiples containing one of the digits 3, 5, or 6. The chains containing one of these three multiples can have length at most 3.

Hence a chain of length 4 or more can only consist of the multiples 14, 21, 28, 42, 49, 84, 91, and 98. First, we check whether it is possible to construct a chain from all eight remaining multiples. Because two of these multiples end with a 1 and there is only one that starts with a 1, the last multiple in the chain must end with a 1. There are also two multiples ending with an 8 and only one multiple starting with an 8. Therefore, the last multiple in the sequence must also end with an 8. The last multiple cannot end with both a 1 and an 8. Hence, we cannot construct a chain of length 8.

It is possible to construct a chain of length 7, for example $21 - 14 - 42 - 28 - 84 - 49 - 91$.

A5. D) 20 The first rule says that the number of black squares in a column cannot equal the number of black squares in a column directly to the right (or left) of it. From the second rule it follows that the number of black squares in a column also cannot equal the number of black squares in a column two positions to the right (or left) of it. Hence, the number of black squares in the first three columns must be different, and as soon as we know these numbers, we also know the numbers of black squares in the other two columns.

The possibilities for the numbers of black squares in the five columns are:

$$0-1-2-0-1, \qquad 0-2-1-0-2, \qquad 1-0-2-1-0,$$

$$1-2-0-1-2, \qquad 2-0-1-2-0, \qquad 2-1-0-2-1.$$

If there are 0 or 2 black squares in a column, then this can be done in only 1 way. If there must be 1 black square in a column, then this can be done in 2 ways.

Hence, the combinations $0-2-1-0-2$ and $2-0-1-2-0$ can each occur in 2 ways, and the other four combinations can each occur in $2 \times 2 = 4$ ways. In total, there are $2 \times 2 + 4 \times 4 = 20$ possibilities.

A6. E) 35 You can obtain 35 as $5 \times (4+3) : (2-1)$. This is the largest number among the possible answers, and therefore the right answer.

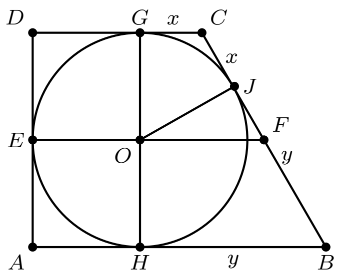
A7. D) Isa has a mountain bike. Suppose that Isa does not have a mountain bike. Then the other statement of Isa must be true and Nick has the electric bike. Hence, Isa has neither the mountain bike, nor the electric bike and must therefore have the racing bike. Because Nick has the electric bike, Agatha's first statement is false; her second statement must be true, which gives us the colour of Isa's racing bike: blue. But then both Nick's statements are false, which is a contradiction. Therefore, Isa must have the mountain bike. Hence, D) is always true.

In the following two situations, exactly one of the statements of each of the three cyclists is true. Therefore, these two situations can occur, but the statements in A), B), C), and E) are not true in all possible situations.

Agatha	electric bike	green
Isa	mountain bike	black
Nick	racing bike	blue

Agatha	electric bike	black
Isa	mountain bike	green
Nick	racing bike	blue

A8. D) 22 We denote the points where the circle touches CD, AB, and BC by G, H, and J, respectively. We denote the centre of the circle by O. Let the length of CG be x and let the length of BH be y.



Since quadrilateral $OJCG$ is mirror symmetric around OC, we see that $|CJ| = x$. In a similar way we find that $|BJ| = y$ because $OHBJ$ is symmetric around OH. We conclude that $x + y = 24$.

The length of AB is equal to $|AH| + |BH| = 10 + y$, and the length of CD is equal to $|CG| + |DG| = 10 + x$. Since EF is located precisely halfway trapezium $ABCD$, the length of EF is equal to the average of the lengths of AB and CD. We therefore find that $|EF| = 10 + \frac{1}{2}(x + y) = 22$.

B1. 45 minutes In one hour, Maurits travels 2 kilometres more at the pace of route B than at the pace of route A. Therefore, it takes him $\frac{1.5}{2} \times 60 = 45$ minutes to travel 1.5 kilometre extra at the fast pace compared to the slower pace. It follows that he takes 45 minutes to bike to school.

B2. 1836 The number n must be a four-digit number. Indeed, a three-digit number added to one of its fragments cannot give a number larger than $999 + 99 < 2019$, and a five-digit number is already larger than 2019 by itself. We write $n = abcd$ where a, b, c, and d are the digits of n. Since we want n to be as small as possible, we should choose a three-digit fragment (if possible). This means that $abcd + bcd = 2019$ or $abcd + abc = 2019$. Since $abcd + bcd$ will always be an even number, this case is ruled out. Hence, we have

$$\begin{array}{cccccc} a & b & c & d & & \\ & a & b & c & & + \\ \hline 2 & 0 & 1 & 9 & & \end{array}$$

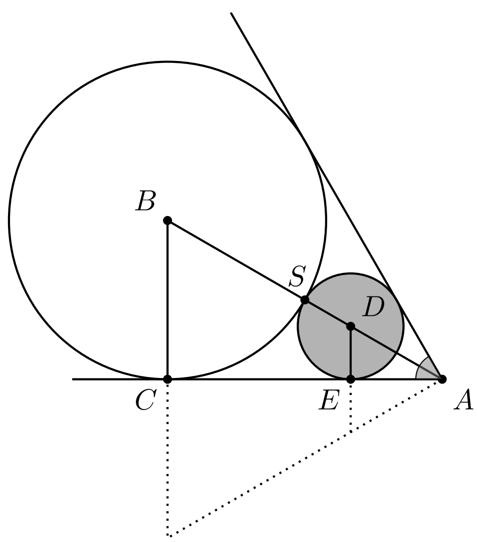
Note that in this addition we may create a carry in certain positions. This is not the case in the rightmost position: $c + d$ is equal to 9, not to 19.

We consider the digits from left to right. We see that $a = 1$ or $a = 2$. The case $a = 2$ does not happen since this would give $b + a = 0$ in the adjacent position. So we have $a = 1$ and we have $b + a = 9$ or $b + a = 10$, hence $b = 8$ or $b = 9$. The second case cannot occur because that would imply that $c + b = 1$ in the adjacent position. We therefore find that $b = 8$ and that $c + b = 11$. This means that $c = 3$. Finally, we consider the rightmost position and see that $d + c = 9$, hence $d = 6$. We conclude that $n = 1836$ which is indeed a solution because $1836 + 183 = 2019$.

B3. $\frac{10}{9}$ In the figure, a magnification of one corner of the equilateral triangle is drawn. The largest circle is not drawn, only the second and the third circle. The two sides meet in vertex A at an angle of 60 degrees. We will show that the second circle has a radius that is three times the radius of the third circle: $|BC| = 3|DE|$.

We first consider triangle ABC. Angle BAC is $\frac{60}{2} = 30$ degrees, angle ACB is 90 degrees, and angle ABC is therefore $180 - 30 - 90 = 60$ degrees. This means that triangle ABC and its reflection in line AC together form an equilateral triangle. It follows that $|AB| = 2|BC|$. The same reasoning for triangle ADE shows that $|AD| = 2|DE|$. Now consider line segment AS. We have $|AS| = |AB| - |BS| = 2|BC| - |BC| = |BC|$. On the other hand, we also have $|AS| = |SD| + |AD| = |DE| + 2|DE| = 3|DE|$. Thus, we obtain $|BC| = 3|DE|$ as claimed.

We can use the same reasoning on the first and second circle to find that the radius of the first circle is three times the radius of the second circle. The radius of the third circle is therefore equal to $\frac{1}{3} \cdot \frac{1}{3} \cdot 10 = \frac{10}{9}$.



B4. 12 There are 9 letters, namely A to I. The number of pairs of letters is $\frac{9\times 8}{2} = 36$. Indeed, each of the 9 letters can form a pair with 8 other letters. This way, we count each pair twice (the pair AE is the same as the pair EA).

Each card has three letters that together make 3 pairs. Therefore, we need at least $\frac{36}{3} = 12$ cards if we want to get all 36 pairs. That 12 cards is sufficient is shown by the following choice of triples:

ABC	DEF	GHI
ADG	BEH	CFI
AEI	CDH	BFG
AFH	BDI	CEG

The triples are also shown in the figure. The twelve triples are depicted by the three horizontal lines, the three vertical lines, the two diagonals, and the four curved lines. It is easy to check that every two letters occur together on one of the lines.