

First round

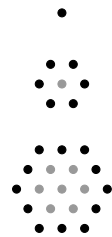
Dutch Mathematical Olympiad

23 January – 2 February 2017

Solutions

- A1.** **C) Thursday** In the first four weeks of August, every day of the week occurs exactly four times. The last $31 - 4 \times 7 = 3$ days of August are the days of the week that occur five times in August. Hence, Monday and Friday cannot be one of those three days. The only remaining possibility is for the last three days to be Tuesday, Wednesday, and Thursday. Therefore, the last day of August must be a Thursday.

- A2.** **D) 169** The smallest dotted hexagon consists of a single dot. To construct the next hexagon, we have to add six new dots around the existing dot. To construct the third hexagon, we again add one layer of dots around the current hexagon. In this case we add twelve new dots. The number of dots we have to add in order to construct the next hexagon increases by six in each step. Indeed, the number of dots along each side of the hexagon increases by one in every step.



After 19, the next hexagonal number is therefore $19 + 18 = 37$. The next ones after that are $37 + 24 = 61$, $61 + 30 = 91$, $91 + 36 = 127$, $127 + 42 = 169$, and then $169 + 48 = 217$. We conclude that of the five given options, only 169 is a hexagonal number.

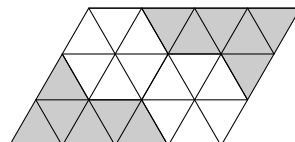
- A3.** **C) 3** At most one of the statements made by Eva, Fatima, and Kees can be true. At least two of them are lying and are therefore guilty. This already implies that Eva and Kees are lying and are guilty. Hence, Mustafa must be telling the truth and is therefore innocent.

Fatima cannot be telling the truth, because then both she and Mustafa would be innocent, contradicting her own statement. So Fatima is guilty.

Manon cannot be lying because then she would be guilty and Mustafa would be the only one that is innocent. This contradicts the established fact that Fatima is lying.

We conclude that Manon and Mustafa are the only ones who are innocent. This is indeed consistent with their own statements and the falsehood of the statements of the other three.

- A4.** **E) $\frac{1}{2}$** We start by dividing each of the two regular hexagons into six equal equilateral triangles. Using another twelve of the same triangles, we can complete this to a tiling of the parallelogram, see the figure. It follows that the two grey areas together are exactly $\frac{12}{24}$ of the area of the parallelogram. Therefore, this area equals $1 \cdot \frac{12}{24} = \frac{1}{2}$.



- A5.** **D) 222** The fraction $\frac{83+74+65}{20-19}$ gives us a value of 222. We will now show that no larger outcome is possible.

First we observe that the numerator must be smaller than $100 + 100 + 100 = 300$. If the denominator is larger than 1, the outcome must therefore be smaller than $\frac{300}{2} = 150$. Hence, to obtain a value of 222 or more, the denominator must be equal to 1.

The only ways to get a denominator equal to 1 is through one of $20 - 19, 30 - 29, \dots, 80 - 79$. In each case, the 9 is used in the denominator, hence the largest remaining digits for the numerator

are 8, 7, 6, 5, 4, and 3. To maximise the numerator, we place the largest three digits in the tens' places. Finally, we put the remaining three digits in the unit places. The value of the fraction will then be $80 + 70 + 60 + 5 + 4 + 3 = 222$.

- A6.** **B) 3** Consider a route from the bottom left cell to the top right cell. This route consists of $99 + 99 = 198$ steps. We denote the numbers in the cells along this route by $a_0, a_1, a_2, \dots, a_{198}$. We know that $a_0 = 0$, $a_1 = 1$, and $a_2 = 3$. To compute the next number, a_3 , we have to add the numbers a_0 , a_1 , and a_2 and add an additional 1 for each step. Thus, we get $a_3 = (a_0 + 1) + (a_1 + 1) + (a_2 + 1) = 7$. To compute the next number, a_4 , we have to compute the exact same sum and then add $a_3 + 1$. We therefore see that $a_4 = a_3 + (a_3 + 1) = 15$. In general, we have the following equation:

$$a_{k+1} = (a_1 + 1) + \dots + (a_{k-1} + 1) + (a_k + 1) = a_k + (a_k + 1) = 2a_k + 1.$$

Considering only the last digits, we notice a repeating pattern; after the first 0, we find 1, 3, 7, 5, 1, 3, 7, 5, and so on. The fact that this repetition occurs is easy to explain. Indeed, the last digit of a_{k+1} only depends on the last digit of a_k . For this reason, the digits repeat in the same pattern, all the way up to a_{198} . We conclude that the numbers $a_2, a_6, a_{10}, a_{14}, \dots, a_{198}$ all have 3 as their last digit.

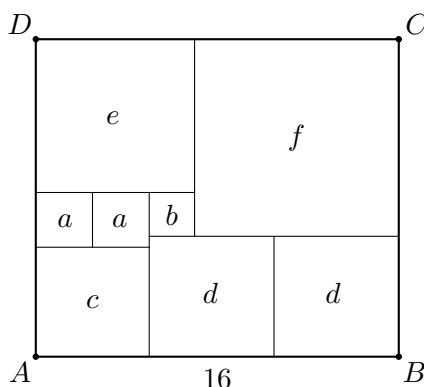
- A7.** **D) $\frac{29}{2}$** In the figure, the side length of each square is indicated by a letter. We see that $2a + 2d = 16$, and therefore $d = 8 - a$. We can similarly express all other side lengths in terms of a through the following relations seen from the figure:

$$\begin{aligned} c &= 2a, \\ b &= a + c - d = 3a - (8 - a) = 4a - 8, \\ e &= 2a + b = 6a - 8, \\ f &= b + e = (4a - 8) + (6a - 8) = 10a - 16. \end{aligned}$$

Of course we also know that $e + f = 16$, which implies that $(6a - 8) + (10a - 16) = 16$. This gives us $16a = 40$, and hence $a = \frac{5}{2}$.

The length AD can now be expressed as

$$|AD| = a + c + e = 3a + (6a - 8) = 9a - 8 = \frac{45}{2} - 8 = \frac{29}{2}.$$



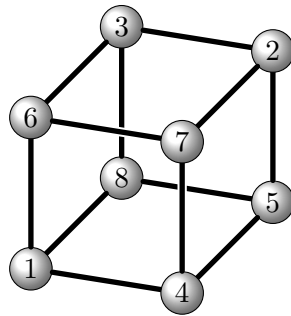
A8. E) the fifth We determine which of the figures can represent Joep's cube.

Two opposite facets of the cube together use all of the eight numbers exactly once. Therefore, the numbers on opposite facets must always add up to $1 + 2 + \dots + 8 = 36$. The first figure is now eliminated since it has 18 and 17 on opposite facets, while $18 + 17 \neq 36$. Similarly, the second figure is ruled out since it has 18 and 20 on opposite facets.

We can rule out the third figure because it is not even a layout of a cube. When folding this figure, the squares containing the numbers 10 and 19 will overlap.

The fact that the number on each facet must be at least $1 + 2 + 3 + 4 = 10$ eliminates the fourth figure.

Finally, we will show that the fifth figure can represent Joep's cube. In the figure below you can see an assignment of the numbers 1 to 8 to the vertices of the cube with the property that for each facet we obtain a sum of 18, as required.



B1. 911 We will consider the possible numbers that Isaac could have written down. We will distinguish three cases: there can occur one, two, or three distinct digits in Isaac's number.

- If only one digit occurs in Isaac's number, then this number is of the form \overline{aaa} (by which we mean the number consisting of three times the digit a). Dilara then writes down no new numbers, because there is no other number which you can get by putting the digits of \overline{aaa} in another order. Hence, the sum is \overline{aaa} which is smaller than 1221 for all possible digits a .
- If only two distinct digits occur in Isaac's number, then this number is of one of the forms \overline{aab} , \overline{aba} or \overline{baa} (with $a \neq b$). If Isaac's number is one of these three numbers, then Dilara writes down the other two. The sum then becomes $\overline{aab} + \overline{aba} + \overline{baa}$. We want this sum to end with the digit 1, hence we want $b + a + a = 11$ or $b + a + a = 21$ to hold. However, if $b + a + a = 21$, then the total sum must become at least 2100 (because the sum of the first digits will also be 21) which is greater than 1221. Therefore, the only possibility is $b + a + a = 11$.

To get a number which is as large as possible, we want its first digit to be as large as possible. We remark that a can be at most 5, but b could be 9 if a were equal to 1. We try $b = 9$ and $a = 1$ and consider the number $\overline{baa} = 911$ for Isaac to start with. Then Dilara writes down the numbers 191 and 119 and we see that the sum $911 + 191 + 119$ indeed equals 1221.

- If Isaac's number contains three distinct digits, then this number is of the form \overline{abc} (with a , b , and c distinct). Dilara then writes down the numbers \overline{acb} , \overline{bac} , \overline{bca} , \overline{cab} , and \overline{cba} . If we now consider these six numbers, then we notice that for the last digit we have two times the digit a , two times the digit b , and two times the digit c . Hence, the last digit of the sum equals the last digit of $2a + 2b + 2c$. This digit, however, is always even, while the last digit of 1221 is odd. Therefore, the sum cannot become equal to 1221.

We have proved that Isaac's number must consist of two distinct digits, and that the largest number he could have started with, in order to obtain the sum 1221, is the number 911.

B2. (13, 2, 8) and (5, 6, 4) We subtract the second equation from the first. This yields

$$ab + c - a - bc = 5.$$

The left-hand side of this equation can also be written as $(a - c)(b - 1)$. Hence, we find $(a - c)(b - 1) = 5$. Because b is a positive integer, the factor $b - 1$ cannot be negative. As 5 is a prime number, there are two possibilities left: $b - 1 = 1$ and hence $b = 2$, or $b - 1 = 5$ and hence $b = 6$.

If $b = 2$, then we also immediately find $a - c = 5$. If we add this to the equation $ab + c = 34$, then we find $a + ab = 39$. As $b = 2$, this yields $a = 13$ and hence $c = 8$. If we substitute $(a, b, c) = (13, 2, 8)$ in the original equations, then we verify that this is indeed a solution.

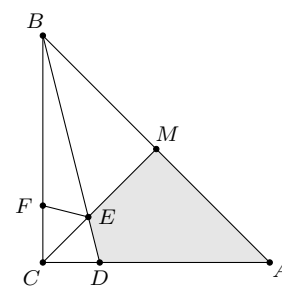
If $b = 6$, then we also immediately find $a - c = 1$. Now we add this to the equation $ab + c = 34$ to find $a + ab = 35$. In this case we had $b = 6$ and hence we find $a = 5$. This also immediately yields $c = 4$. If we substitute $(a, b, c) = (5, 6, 4)$ in the original equations, then we verify that this is another solution.

To summarize, the two solutions are $(a, b, c) = (13, 2, 8)$ and $(a, b, c) = (5, 6, 4)$.

- B3.** $\frac{162}{5}$ The area of $\triangle AMC$ is half the area of $\triangle ABC$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot |AC| \cdot |BC| = \frac{1}{2} \cdot 12 \cdot 12 = 72$. Hence, the area of $\triangle AMC$ is 36. In order to determine the area of quadrilateral $AMED$ we will now determine the area of $\triangle CDE$.

Let F be the point obtained from D by reflection in the line CE . Because of symmetry, $\triangle CEF$ and $\triangle CDE$ have equal areas. Moreover, we also have $|CF| = 3$ and $|BC| = 12$. Triangles BCE and CEF have equal heights, but the basis of $\triangle BCE$ is four times as large. Hence, the area of $\triangle BCE$ is four times the area of $\triangle CEF$ (or $\triangle CDE$).

If we add triangle CDE to this, we get that $\triangle BCD$ has an area five times as large as $\triangle CDE$. The area of $\triangle BCD$ is $\frac{1}{2} \cdot |CD| \cdot |BC| = \frac{1}{2} \cdot 3 \cdot 12 = 18$. Altogether, we find that the area of $\triangle CDE$ equals $\frac{18}{5}$, and that the area of quadrilateral $AMED$ therefore equals $36 - \frac{18}{5} = \frac{162}{5}$.



- B4.** 27 To simplify the problem, we will consider a variation to the quiz. For each question, we give one additional point: we award 6 points for a correct answer to a difficult question, 4 points for a correct answer to an easy question, and 0 points for a wrong answer. Thus, all scores are increased by 10 points, not altering the number of possible final scores. With this modification, all scores are divisible by 2, and we will divide all scores by 2 to simplify the problem even more. A contestant can now get 3, 2, or 0 points for each question. Again, this does not change the number of possible final scores.

The final score is at most 30 points (all questions are answered correctly), and at least 0 (all questions are answered incorrectly). Next, we will show that the final scores 1, 25, 28, and 29 are impossible to obtain and that all other scores from 0 to 30 can be obtained.

The final scores 0, 3, 6, \dots , 30 can be obtained by first, respectively, answering 0, 1, 2, 3, \dots , 10 questions correctly and then answering all other questions incorrectly.

The final scores 2, 5, 8, \dots , 26 can be obtained by first, respectively, answering 9, 8, 7, 6, \dots , 1 questions incorrectly, then answering one easy question correctly and all following difficult questions correctly.

The final scores 4, 7, 10, \dots , 22 can be obtained by first, respectively, answering 7, 6, 5, \dots , 1 questions incorrectly, then answering an easy question correctly, a difficult question incorrectly, an easy question correctly, and all following difficult questions correctly.

If at least one question is answered correctly, then you already have at least 2 points. For this reason, a final score of 1 is impossible to obtain. If at least one question is answered incorrectly, then you can have at most $9 \cdot 3 = 27$ points. For this reason, final scores of 28 and 29 are impossible to obtain. If you answered at least one question incorrectly, then two things could have happened: the incorrectly answered question was the last question and you obtained $9 \cdot 3 = 27$ points, or the incorrectly answered question was not the last question and you obtained $8 \cdot 3 + 2 = 26$ points. If you answered at least two questions incorrectly, then you got at most $8 \cdot 3 = 24$ points. For this reason, a final score of 25 is also impossible to obtain.

To summarise, the final scores possible to obtain (in the new quiz format) are the numbers from 0 to 30, except 1, 25, 28, and 29. (In the original quiz format, the final scores possible to obtain are the even numbers from -10 up to and including 50, except -8 , 40, 46 and 48.) Therefore, there are 27 possible final scores.