60th Dutch Mathematical Olympiad 2021

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2022 started in The Netherlands with a first round in January 2021, held locally at all participating schools. The paper consists of eight multiple choice questions and four open questions, to be solved within 2 hours. This year 3403 students from 229 secondary schools participated in this 60th edition of the Dutch Mathematical Olympiad.

The 734 best students were invited to the second round, which was held at twelve universities throughout the country a few months later, March 2021. This round consists of five open questions, and two problems for which the students have to give extensive solutions and proofs. The contest lasts 2.5 hours.

The 138 best students were invited to the final round, with the addition of some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. The final is preceded by up to four training sessions at the universities to help them prepare for their participation in the final round.

The final, in September, contains five problems for which the students has to give extensive solutions and proofs. They are allowed 3 hours for this round. After the prizes are awarded in the beginning of November, the Dutch Mathematical Olympiad is concluded.

The 30 most outstanding candidates of the Dutch Mathematical Olympiad are invited to an intensive seven-month training programme. The students meet twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. They also work on weekly problem sets to be sent in to a personal trainer by email.

In February a team of four girls was chosen from the training group to represent the Netherlands in April at the EGMO in Hungary. At this event (which was the first on site one in quite a while) the Dutch team won one silver medal and an honourable mention. For more information about the EGMO (including the 2022 paper), see www.egmo.org.

In March a selection test of 3.5 hours is held to determine the ten students from the training program which are sent to the Benelux Mathematical Olympiad (BxMO) held in May. In a historic event each of the students of the Dutch team 2022 managed to take home the honours: two gold medals, five silver medals and three bronze medals. For more information about the BxMO (including the 2022 paper), see www.bxmo.org.

Begin June the team for the International Mathematical Olympiad is selected by three team selection tests on three consecutive days, each lasting 4 hours. In addition to the six team members a seventh, young, promising student is selected to accompany the team to the IMO as an observer C. Three weeks later, the team had a training camp north of Oslo from July 1 till 9.

For younger students the Junior Mathematical Olympiad was held in September 2021 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2022 consists of

- Lance Bakker (15 years old)
  - silver medal at BxMO 2022
- Jelle Bloemendaal (18 years old)
  - bronze medal at BxMO 2019, silver medal at BxMO 2020 and 2021
  - (virtual) observer C at IMO 2020, bronze medal at IMO 2021
- Mads Kok (15 years old)
  - silver medal at BxMO 2022
- Casper Madlener (17 years old)
  - silver medal at BxMO 2020, gold medal at BxMO 2022
  - (virtual) observer C at IMO 2020, participant at IMO 2021
- Lars Pos (18 years old)
  - bronze medal at BxMO 2021, gold medal at BxMO 2022
  - (virtual) observer C at IMO 2021
- Kees den Tex (18 years old)
  - participant at BxMO 2020, gold medal at BxMO 2021, silver medal at BxMO 2022
  - honourable mention at IMO 2021

Also part of the IMO selection, but not officially part of the IMO team, is:

- Wouter Zandsteeg (17 years old)
  - bronze medal at BxMO 2022

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Johan Konter (deputy leader), Leiden University
- Ward van der Schoot (observer B), Applied Cryptography and Quantum Applications, TNO

First Round, January/February 2021

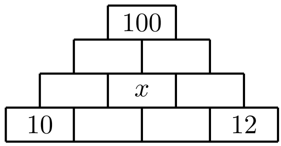
Problems

A-problems

1. For the integers a, b, c, and d the difference between a and b equals 2, the difference between b and c equals 3, and the difference between c and d equals 4. Which of the following values <u>cannot</u> be the difference between a and d?

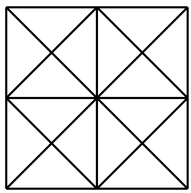
A) 1 B) 3 C) 5 D) 7 E) 9

2. In each square of the top three rows in the pyramid on the right, the number written in that square equals the sum of the numbers in the two squares below it. For three of the squares, the numbers written in them are given.
What number must be written in the square with the x in it?



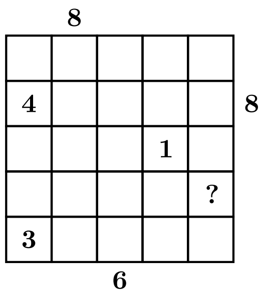
A) 17 B) 20 C) 23 D) 26 E) 39

3. How many triangles are there in the figure on the right?



A) 32 B) 36 C) 40 D) 44 E) 64

4. In each square of the field on the right, there is a high-rise building of height 1, 2, 3, 4, or 5, such that the following conditions are satisfied.



- In each (horizontal) row or (vertical) column, each height occurs exactly once.
- The numbers on the side of the square are the sums of the heights of the visible buildings. This concerns the buildings in this particular row or column that are (partially) visible in the side view from the number on the side. For example, if the heights 1, 3, 2, 5, and 4 occur in this order in a row, then the buildings of heights 1, 3, and 5 are visible from the left side, and the buildings of heights 4 and 5 are visible from the right side.

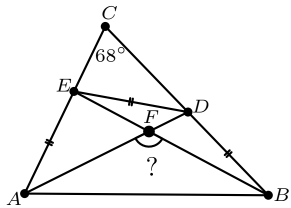
What is the height of the building on the square with the question mark?

A) 1 high B) 2 high C) 3 high D) 4 high E) 5 high

5. The number 1 is written on the blackboard. A *turn* consists of wiping out the number on the board and replacing it by the double of the number, or by the number one smaller. For example, we can replace 1 by 2 (the double) or 0 (one smaller), and if 5 is on the board, we can replace it by 10 or 4.
What is the minimum number of turns needed in order to write the number 2021 on the board?

A) 14 B) 15 C) 16 D) 17 E) 18

6. In triangle ABC, a point D lies on side BC and a point E lies on side AC such that the line segments BD, DE, and AE have the same length. The point F is the intersection between the line segments AD and BE. Angle C is $68°$.
What is the size of angle F in triangle AFB?



A) $120°$ B) $121°$ C) $122°$ D) $123°$ E) $124°$

7. The integers 1 to n are written on the board. One of the numbers is wiped out. The average of the remaining numbers is $11\frac{1}{4}$.
Which number has been wiped out?

A) 6 B) 7 C) 11 D) 12 E) 21

8. We order the positive odd integers as follows:

	column 1	column 2	column 3	column 4	column 5	column 6	...
row 1	1	3	11	13	29	31	...
row 2	5	9	15	27	33	...	
row 3	7	17	25	35	...		
row 4	19	23	37	...			
row 5	21	39	...				
row 6	41	...					
...	...						

For each odd number we can determine in which row and column it is placed. For example, the number 35 is placed in row 3 and column 4.
What number is placed in row 22 and column 24?

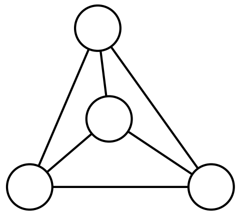
A) 2021   B) 2023   C) 2025   D) 2027   E) 2029

B-problems

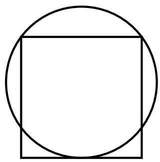
The answer to each B-problem is a number.

1. We have two integers consisting of two digits, and both numbers do not start with a 0. If you add these numbers, you get the number S. If you interchange the two digits of both numbers and add the new numbers, you get $4S$.
Determine all possible pairs of two-digit numbers satisfying these constraints. Make sure to clearly indicate in your answer which numbers form a pair.

2. In the diagram on the right we write a number in each circle. The numbers do not have to be integers or be positive. Next to each line segment, we write the sum of the two numbers in the circles on the end of the line segment. There are two quadruples of numbers that we can write in the circles, such that the numbers next to the line segments are exactly the numbers 0, 1, 2, 3, 4, and 5. For both of these quadruples, we multiply the four numbers in the circles with each other.
Which two results can we get from this multiplication?



3. A circle of radius 1 and a square are given, such that the circle is tangent to one side of the square and also two of the vertices of the square lie on the circle.
What is the length of a side of the square?



4. We consider security codes consisting of four digits. We say that one code *dominates* another code if each digit of the first code is at least as large as the corresponding digit in the second code. For example, 4961 dominates 0761, because $4 \geq 0$, $9 \geq 7$, $6 \geq 6$, and $1 \geq 1$. We would like to assign a colour to each security code from 0000 to 9999, but if one code dominates another code then the codes cannot have the same colour.
What is the minimum number of colours that we need in order to do this?

Solutions

A-problems

1. D) 7

2. D) 26

3. D) 44

4. B) 2

5. B) 15

6. E) $124°$

7. A) 6

8. D) 2027

B-problems

1. $\{14, 19\}, \{15, 18\}$, and $\{16, 17\}$

2. -6 and $-\frac{21}{16}$

3. $\frac{8}{5}$

4. 37

Second Round, March 2021

Problems

B-problems

The answer to each B-problem is a number.

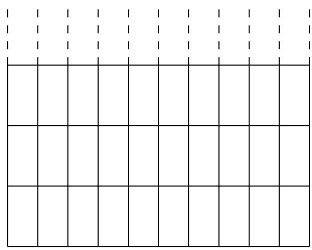
B1. Peter gets bored during the lockdown, so he decides to write numbers the whole day. He makes a sequence of numbers starting with 0, 1 and -1, and then going on indefinitely. On the next line he writes the same sequence of numbers, but shifted one place to the right. On the third line he writes again the same sequence of numbers, shifted another place to the right. He adds all three numbers standing in a vertical column. (He skips the first two places so he starts with $-1+1+0$.) The answer for every column is the next multiple of three. Peter's paper hence looks like this:

$$
\begin{array}{cccccccc}
 & 0 & 1 & -1 & \dots & \dots & \dots & \dots \\
 & & 0 & 1 & -1 & \dots & \dots & \dots \\
+ & & & 0 & 1 & -1 & \dots & \dots \\
\hline
 & & & 0 & 3 & 6 & 9 & 12
\end{array}
$$

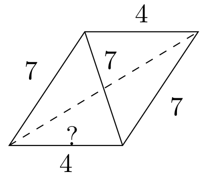
The first number in the uppermost sequence is 0, the second number is 1, the third number is -1, etcetera. Determine the 2021st number in the uppermost sequence.

B2. An integer n is a *combi number* if each pair of distinct digits from the set of all possible digits 0 to 9 appear at least once in the number as neighbouring digits. For example, in a combi number the digits 3 and 5 have to appear somewhere next to each other. It does not matter whether they appear in the order 35 or 53. We take the convention that a combi number never starts with the digit 0.
What is the smallest possible number of digits of a combi number?

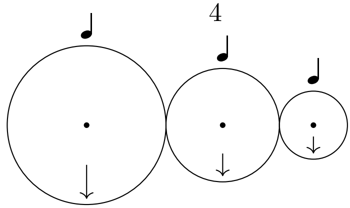
B3. A big rectangle is divided in small rectangles that are twice as high as they are wide. The rectangle is 10 of these small rectangles wide, as in the figure on the right. In this figure you can see some squares of different sizes.
How many small rectangles high is the figure if we can find exactly 345 squares in it?



B4. A parallelogram has two sides of length 4 and two sides of length 7. Also, one of the diagonals has length 7. (Attention: the picture has not been drawn to scale.) What is the length of the other diagonal?



B5. Three wheels are pushed together so they don't slip if we turn them. The circumferences of the wheels are 14, 10, and 6 cm, respectively. On each wheel an arrow is drawn, pointing downwards. Someone turns the big wheel and the other wheels turn with it. This stops at the first moment all arrows point downwards again. Every time one of the arrows is pointing up, a whistle sounds. If two or three arrows point up at the same time, only one whistle sounds. How many whistles sound in total?



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. Around a round table $n \geq 3$ players are sitting. The game leader divides n coins among the players, in such a way that not everyone gets exactly one coin. Any player can see the number of coins of each other player.
Every 10 seconds, the game leader rings a bell. At that moment, each player looks how many coins their two neighbours have. Then they all do the following at the same time:

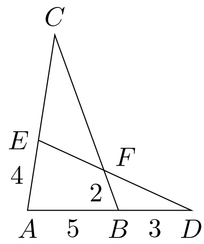
- If a player has more coins than at least one of their neighbours, the player gives away exactly one coin. They give this coin to the neighbour with the smallest number of coins. If both of their neighbours have the same number of coins, they give the coin to the neighbour on the left.
- If a player does not have more coins than at least one of their neighbours, the player does nothing and waits for the next round.

The game ends if everyone has exactly one coin.

(a) For each $n \geq 3$, find a distribution of the coins at the start such that the game will never stop (and prove that the game does not stop for your starting distribution).

(b) For each $n \geq 4$, find a distribution of the coins at the start of the game such that the game will stop (and prove that the game stops for your starting distribution).

C2. We consider a triangle ABC and a point D on the extended line segment AB on the side of B. The point E lies on side AC such that the angles $\angle DBC$ and $\angle DEC$ are equal. The intersection of DE and BC is F. Suppose that $|BF| = 2$, $|BD| = 3$, $|AE| = 4$, and $|AB| = 5$. (Attention: the picture has not been drawn to scale.)



(a) Prove that triangles $\triangle ABC$ and $\triangle AED$ are similar.

(b) Determine $|CF|$.

Solutions

B-problems

1. 2020

2. 50

3. 15

4. 9

5. 5

C-problems

C1. (a) Consider the situation where the first player has 2 coins, the second player has 0 coins and all other players have 1 coin. This situation looks as follows:

$$2\,0\underbrace{1\,1\cdots1\,1}_{n-2 \text{ ones}}$$

For example, for $n = 3$ *the starting distribution is* 201*. We see that the first and the third player both give a coin to the second player. This gives the distribution* 120*. This is exactly the distribution* 201 *if you shift all players by one place. We see that the game never stops.* In this case the first player has to give a coin to the second player, the third player has to give a coin to the left and all other players keep their coin. We end with the following situation.

$$1\,2\,0\underbrace{1\,1\cdots1\,1}_{n-3 \text{ ones}}$$

This is exactly the same distribution as the starting distribution, except now it is player 2 that has 2 coins and player 3 that has 0 coins. If we continue playing, there will always be a player with 2 coins and thus the game never stops. $\square$

(b) For $n \geq 4$ we can consider the following starting distribution.

$$2\,0\,0\,2\underbrace{1\,1\cdots1\,1}_{n-4 \text{ ones}}$$

For example, for $n = 4$ the starting distribution is 2002. In this case there is no player with exactly one coin. The first and the last player give a coin to the second and third player, respectively. Then the game stops.

The first player has to give a coin to the right and the fourth player has to give a coin to the left. All other players keep their coin. This gives a situation where all players have 1 coin, thus the game stops. □

C2. (a) Because angles $\angle AEC$ and $\angle ABD$ are straight, we have

$$\angle ABC = 180° - \angle DBC = 180° - \angle DEC = \angle AED.$$

Because angle A occurs in both triangles, triangles $\triangle ABC$ and $\triangle AED$ have two equal angles, and hence the triangles are similar. □

(b) Because of the similarity of triangles $\triangle ABC$ and $\triangle AED$, the angles at C and D are equal. Together with the equality $\angle DBF = \angle CEF$, it follows that triangles $\triangle DBF$ and $\triangle CEF$ are similar.

In a pair of similar triangles, all pairs of sides have the same ratio. Hence, the similarity of triangles $\triangle DBF$ and $\triangle CEF$ yields

$$\frac{|BF|}{|EF|} = \frac{|FD|}{|FC|} = \frac{|DB|}{|CE|}. \tag{1}$$

As triangles $\triangle ABC$ and $\triangle AED$ are similar, we find that

$$\frac{|AB|}{|AE|} = \frac{|BC|}{|ED|} = \frac{|CA|}{|DA|}. \tag{2}$$

Using equations (1) and (2), we can now find $|CF|$. Using the first and last ratio in equation (2), we get $\frac{5}{4} = \frac{|AB|}{|AE|} = \frac{|AC|}{|AD|} = \frac{4+|EC|}{5+3}$. Hence, we have $|EC| = 6$. If we substitute this in the first and third ratio in equation (1), we get $\frac{2}{|EF|} = \frac{3}{6}$. Hence, we have $|EF| = 4$. Using the first and second ratio in (1), we now get that $\frac{2}{4} = \frac{|FD|}{|FC|}$ hence $|FD| = \frac{1}{2}|CF|$. Finally, we substitute this in the first and second ratio in equation (2):

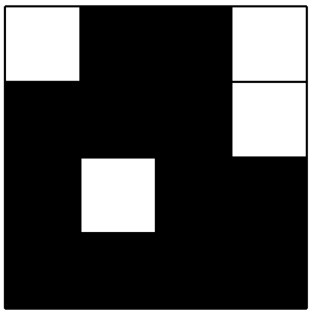
$$\frac{5}{4} = \frac{|AB|}{|AE|} = \frac{|BC|}{|DE|} = \frac{2+|CF|}{4+\frac{1}{2}|CF|}.$$

Taking cross ratios and solving the remaining equation, we get $|CF| = 8$. □

Final Round, September 2021

Problems

1. Niek has 16 square cards that are white on one side and black on the other. He puts down the cards to form a 4×4-square. Some of the cards show their white side and some show their black side. For a colour pattern he calculates the *monochromaticity* as follows. For every pair of adjacent cards that share a side he counts $+1$ or -1 according to the following rule: $+1$ if the adjacent cards show the same colour, and -1 if the adjacent cards show different colours. Adding this all together gives the monochromaticity (which might be negative). For example, if he lays down the cards as below, there are 15 pairs of adjacent cards showing the same colour, and 9 such pairs showing different colours.



The monochromaticity of this pattern is thus $15 \cdot (+1) + 9 \cdot (-1) = 6$. Niek investigates all possible colour patterns and makes a list of all possible numbers that appear at least once as a value of the monochromaticity. That is, Niek makes a list with all numbers such that there exists a colour pattern that has this number as its monochromaticity.

(a) What are the three largest numbers on his list?
(Explain your answer. If your answer is, for example, 12, 9 and 6, then you have to show that these numbers do in fact appear on the list by giving a colouring for each of these numbers, and furthermore prove that the numbers 7, 8, 10, 11 and all numbers bigger than 12 do not appear.)

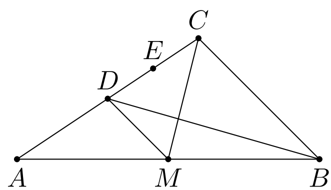
(b) What are the three smallest (most negative) numbers on his list?

(c) What is the smallest positive number (so, greater than 0) on his list?

2. We consider sports tournaments with $n \geq 4$ participating teams and where every pair of teams plays against one another at most one time. We call such a tournament *balanced* if any four participating teams play exactly three matches between themselves. So, not all teams play against one another.
Determine the largest value of n for which a balanced tournament with n teams exists.

3. A frog jumps around on the grid points in the plane, from one grid point to another. The frog starts at the point $(0, 0)$. Then it makes, successively, a jump of one step horizontally, a jump of 2 steps vertically, a jump of 3 steps horizontally, a jump of 4 steps vertically, et cetera. Determine all $n > 0$ such that the frog can be back in $(0, 0)$ after n jumps.

4. In triangle ABC we have $\angle ACB = 90°$. The point M is the midpoint of AB. The line through M parallel to BC intersects AC in D. The midpoint of line segment CD is E. The lines BD and CM are perpendicular.



Be aware: the figure is not drawn to scale.

(a) Prove that triangles CME and ABD are similar.

(b) Prove that EM and AB are perpendicular.

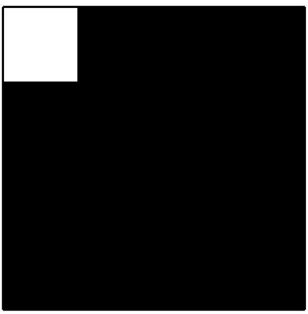
5. We consider an integer $n > 1$ with the following property: for every positive divisor d of n we have that $d + 1$ is a divisior of $n + 1$. Prove that n is a prime number.

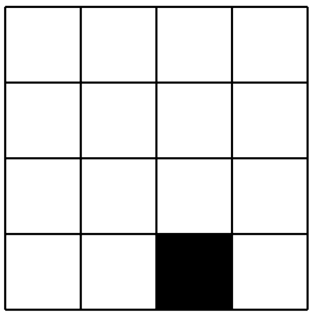
Solutions

1. (a) First note that there are $3 \cdot 4 = 12$ horizontal borders between two cards, and also 12 vertical borders. Suppose that k of these borders count as -1, then there are $24 - k$ borders counting as $+1$. This gives a monochromaticity of $(24 - k) \cdot (+1) + k \cdot (-1) = 24 - 2k$. Hence, the monochromaticity is always an even number.

If all cards have the same colour, then all borders count as $+1$, and we get the maximal monochromaticity of 24. Can 22 also occur as the monochromaticity? No, and we will prove that by contradiction. Suppose there is an assignment of cards having monochromaticity 22. Then there has to be one border with -1 and the rest must count as $+1$. In other words, all adjacent cards have the same colour, except for one border. Consider the two cards at this border, and choose two adjacent cards so that you obtain a 2×2 square. For each pair of cards, you can find such a 2×2 square. If you start on the left top and go around the four cards in a circle (left top – right top – right bottom – left bottom – left top), then you cross four borders. Since you are starting and ending in the same colour, you must have crossed an even number of borders where the colour is changing. This, however, is in contradiction with the assumption that there is only one border at which the two cards have different colours. We conclude that the monochromaticity can never be 22.

The next possibilities for large monochromaticities are 20 and 18. Then there have to be 2 or 3 borders between cards of different colours. This can be achieved by the following colourings:

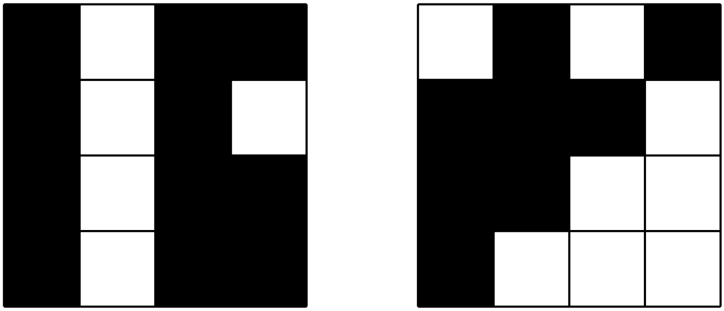




The three largest numbers on Niek's list are 24, 20, and 18. □

(b) Suppose that we put the cards such that the monochromaticity is x. Then we can turn half of the cards, as in a chess board pattern: we turn a card if and only if all of the adjacent cards are not turned. With this operation all borders between cards change sign, and we obtain a monochromaticity of $-x$. In other words, x is a possible value for the monochromaticity if and only if $-x$ is possible. Therefore, the three smallest numbers on Niek's list are the negatives of the three greatest numbers: -24, -20, and -18. □

(c) We already proved that the monochromaticity is always an even number. The smallest possible positive even number is 2. This monochromaticity can be obtained by having 13 borders between squares of the same colour, and 11 borders between squares of different colours. There are many ways to achieve this, for example:



□

2. We will show that 5 is the largest value of n for which a balanced tournament with n teams exists. First we will show that in a balanced tournament with $n \geq 5$ teams, there are no three teams that all play against one another in the tournament.

Suppose towards a contradiction that we can find three teams in a balanced tournament that all play against each other, say teams A, B and C. Because $n \geq 5$ there are two other teams, say D and E. Since A, B and C already play three matches between them, there are no other matches between the quadruple A, B, C and D. In other words: D does not play against A, B and C. The same holds for team E. If we now consider the quadruple A, B, D and E we see that there are at most two matches: A against B, and possibly D against E. This means that we have found four teams such that there are not exactly three matches between these four teams. This is a contradiction.

Now we will show that a balanced tournament is not possible with $n \geq 6$ teams. Suppose that $n \geq 6$ and, towards a contradiction, that a balanced tournament with n teams exists. We look at the first six teams, say teams A to F. Suppose that A plays against at most two of these teams, say at most against B and C but not against D, E and F. Since three matches have to be played among the quadruple A, D, E and F, the teams D, E and F all have to play against one another. This is in contradiction with our previous findings.

We conclude that A has to play against at least three of the teams, for example B, C and D. This gives three matches in the quadruple A, B, C, D, so B, C and D do not play any matches between them. Because the quadruple B, C, D, E also has to play three matches, E has to play against all of B, C and D. But now we find a contradiction in the quadruple A, B, C, E: there are already four matches between these teams (A against B, A against C, B against E, and C against E). Therefore a balanced tournament with $n \geq 6$ does not exist.

To conclude, we will show that a balanced tournent with five teams exists. To make such a tournament, imagine the teams are standing in a circle. Two teams play against each other if they are standing next to each other in the circle. If we look at any quadruple of teams, we see there are exactly three pairs of teams standing next to each other in the circle. So the four teams plays three matches between them. We conclude that 5 is the largest value of n for which a balanced tournament with n teams exists. $\square$

3. We solve this problem in two steps. In part (a), we consider a frog that is jumping only on the (horizontal) line. The frog is making a jump of size 1 to the left or right, a jump of size 2 to the left or right, a jump of size 3 to the left or right, et cetera. We figure out for which n the frog can return to the number 0 after n jumps. This is then used in part (b) to show for which n a frog jumping both horizontally and vertically, can can return to the origin $(0,0)$ after n jumps.

(a) When the frog has finished n jumps, it has made $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ steps in total. To get back at 0, the frog must make the same number of steps to the left and the right. Thus, the total number of steps must be even. This means that $\frac{1}{2}n(n+1)$ is even, and hence $n(n+1)$ is a multiple of four. This yields that n or $n+1$ must be a multiple of four, that is, n is of the form $n = 4k-1$ or $n = 4k$. Potential values for n are $3, 4, 7, 8, 11, 12, \ldots$. Now we will show that for each of these values of n the frog can get back at 0 after n jumps.

We will prove this by induction. For $n = 3$ and $n = 4$, it is not hard to find a solution: $1+2-3=0$ and $1-2-3+4=0$. Now suppose that we can choose pluses and minuses such that $\pm 1 \pm 2 \pm \cdots \pm m = 0$ for a certain integer m. Then we can also find a combination of pluses and minuses such that $\pm 1 \pm 2 \pm \cdots \pm (m+4) = 0$. Indeed:

$$\begin{aligned} &\pm 1 \pm 2 \pm \cdots \pm m + (m+1) - (m+2) - (m+3) + (m+4) \\ &\quad = 0 + (m+1) - (m+2) - (m+3) + (m+4) \\ &\quad = 1 - 2 - 3 + 4 \\ &\quad = 0. \end{aligned}$$

It follows that the frog can indeed get back to 0 after n jumps for each n of the form $n = 4k - 1$ or $n = 4k$. $\square$

(b) This problem actually consists of two variants on part (a), namely in the horizontal and the vertical direction. We start by considering the vertical direction. The frog is making jumps consisting of even numbers of steps. This is actually what was happening in part (a), except that the jumps are twice as long. Hence, the frog can end up on the x-axis if the last jump in the vertical direction consists of $8k-2$ or $8k$ steps. Now we have to investigate whether the frog can also arrive back on the y-axis, and hence at the origin $(0,0)$. The last horizontal jump is one before or one after the last vertical jump, hence the last horizontal jump must consist of $8k-3$, $8k-1$, or $8k+1$ steps.

We will investigate whether it is possible that $\pm 1 \pm 3 \pm \cdots \pm n = 0$ for n of the form $8k-3$, $8k-1$, or $8k+1$. To get back to the y-axis, the total number of horizontal steps must be even. Because each jump consists of an odd number of steps, the frog must make an even number of jumps in the horizontal direction. If the last horizontal jump is of the form $n = 8k-3$ or $n = 8k+1$, then the total number of horizontal jumps is odd. This cannot happen. For the remaining case $n = 8k-1$, we will use induction to prove that this case is possible.

Suppose that the last horizontal jump consists of $8k-1$ steps. We show that we can put pluses and minuses such that $\pm 1 \pm 3 \pm \cdots \pm (8k-1) = 0$. For $k = 1$, we find $1 - 3 - 5 + 7 = 0$. Suppose that for a certain $j \geq 1$, we have $\pm 1 \pm 3 \pm \cdots \pm (8j-1) = 0$. Then

$$\begin{aligned} &\pm 1 \pm 3 \pm \cdots \pm (8j-1) + (8j+1) - (8j+3) - (8j+5) + (8j+7) \\ &\quad = 0 + 1 - 3 - 5 + 7 \\ &\quad = 0 \end{aligned}$$

and we can choose pluses and minuses such that

$$\pm 1 \pm 3 \pm \cdots \pm (8(j+1) - 1) = 0.$$

Now we have shown that we can put pluses and minuses such that $\pm 1 \pm 3 \pm \cdots \pm (8k-1) = 0$ for each integer k.

We conclude that there are two possibilities for the frog to end at the origin $(0,0)$. The first is for $n = 8k-1$: the second to last jump consists of $8k-2$ vertical steps, and the last jump consists of $8k-1$ horizontal steps. The second is $n = 8k$, then the second last jump consists of $8k-1$ horizontal steps and the last jump consists of $8k$ vertical steps. $\square$

4. (a) We first prove the similarity $\triangle CMD \sim \triangle ABC$. Since BC and MD are parallel, we find that $\angle ADM = \angle ACB = 90°$ and also $\angle AMD = \angle ABC$. It follows that $\triangle ABC \sim \triangle AMD$. Because $|AB| = 2|AM|$ we also have that $|AC| = 2|AD|$ and thus $|AD| = |DC|$. This implies the congruence $\triangle AMD \cong \triangle CMD$: both triangles have a right angle at D and the two adjacent sides have the same length. Now we have that $\triangle ABC \sim \triangle AMD \cong \triangle CMD$, and so it holds that $\triangle CMD \sim \triangle ABC$.

Now we will prove that $\triangle CME \sim \triangle ABD$. We already know that $\angle ECM = \angle DCM = \angle CAB = \angle DAB$, and also that

$$\frac{|EC|}{|CM|} = \frac{\frac{1}{2}|DC|}{|CM|} = \frac{\frac{1}{2}|CA|}{|AB|} = \frac{|DA|}{|AB|}.$$

This implies that $\triangle CME \sim \triangle ABD$: the triangles have one equal angle and the two adjacent sides have the same ratio.

(b) Let F be the intersection of BD en CM. Since BD is perpendicular to CM we have that $\angle BFM = 90°$. So in the triangle $\triangle BFM$ we have that $\angle BMF + \angle FBM = 90°$. Because of the similar triangles in part (b) we have $\angle FBM = \angle ABD = \angle CME = \angle FME$. It follows that $\angle BMF + \angle FME = 90°$, hence EM is perpendicular to AB. $\square$

5. Suppose by contradiction that n is not prime. Now consider the greatest divisor $d < n$ of n. Then we can write n as de. Since n is not prime, we have $d > 1$ and hence also $e < n$. Now e must satisfy $e > 1$ and $e \leq d$ (because d is the greatest divisor satisfying $d < n$). Now $d+1$ must be a divisor of $n+1$. Moreover, $d+1$ is a divisor of $(d+1)e = de+e = n+e$. This means that $d+1$ must also be a divisor of the difference $n+e-(n+1) = e-1$. This, however, is impossible, because $e-1$ is a number between 1 and $d-1$. Therefore, our assumption that n is not prime must be false, and n must actually be a prime number. $\square$

BxMO Team Selection Test, March 2022

Problems

1. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ for which $f(n) \mid f(m) - n$ if and only if $n \mid m$ for all natural numbers m and n.

2. Let ABC be an acute triangle, and let D be the foot of the altitude from A. The circle with centre A passing through D intersects the circumcircle of triangle ABC in X and Y, in such a way that the order of the points on this circumcircle is: A, X, B, C, Y. Show that $\angle BXD = \angle CYD$.

3. Find all pairs (p, q) of prime numbers such that

$$p(p^2 - p - 1) = q(2q + 3).$$

4. Given positive real numbers $a_1, a_2, \dots, a_n$ with $n \geq 2$ such that $a_1 a_2 \cdots a_n = 1$, prove that

$$\left(\frac{a_1}{a_2}\right)^{n-1} + \left(\frac{a_2}{a_3}\right)^{n-1} + \ldots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1} + \left(\frac{a_n}{a_1}\right)^{n-1} \geq a_1^2 + a_2^2 + \ldots + a_n^2$$

and determine when equality holds.

5. At a fish market there are 10 stalls, each selling the same 10 kinds of fish. Each fish was caught in either the North Sea or the Mediterranean Sea, and each stall has, for each kind of fish, only fish of one origin. A number, say k, of customers buy exactly one fish from each stall, in such a way that they obtain exactly one of each kind of fish. Moreover, for each pair of customers, there is a kind of fish of which the customers have fish of different origin. Consider all possible ways to supply the stalls according to the rules above.

What is the largest possible value of k?

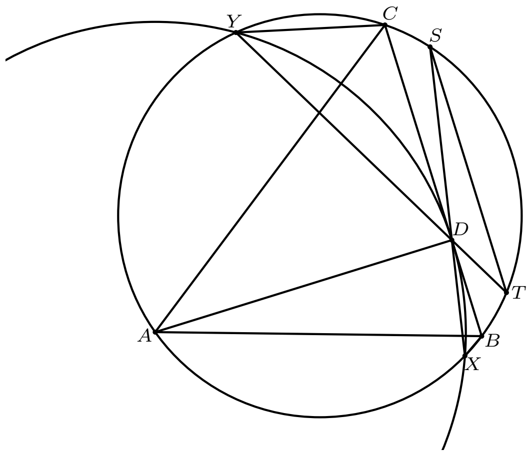
Solutions

1. Substituting $m = n$ gives $f(n) \mid f(n) - n$, so for all natural numbers n we have $f(n) \mid n$. Applying this to the original condition, it follows that $f(n) \mid f(m)$ if and only if $n \mid m$.

We show that $f(n) = n$ by induction on the number of prime factors of n. The base of the induction is the case $n = 1$. In this case, we have $f(1) \mid 1$, therefore $f(1) = 1$.

Suppose that $f(k) = k$ for all natural numbers k with fewer prime factors than n, and suppose for a contradiction that $f(n) \mid n$ is a strict divisor of n. Then there exists a prime number p such that $f(n) \mid \frac{n}{p} = f(\frac{n}{p})$, by the induction hypothesis. But n does not divide $\frac{n}{p}$, which contracts our assumption that $f(n) \neq n$. Therefore $f(n) = n$, and this concludes the induction.

Note that $f(n) = n$ is indeed a solution, since $n \mid m$ holds if and only if $n \mid m - n$. $\square$



2. As the radius AD is perpendicular to BC, the line BC is tangent to the circumcircle of $\triangle DXY$. By the inscribed angle theorem (tangent case), we have $\angle XDB = \angle XYD$. Moreover, the quadrilateral $BCYX$ is cyclic, so $\angle CBX + \angle XYC = 180°$. By the sum of angles in $\triangle BDX$, we have $\angle BXD = 180° - \angle DBX - \angle XDB = (180° - \angle CBX) - \angle XDB = \angle XYC - \angle XYD$. As $\angle XYC - \angle XYD = \angle DYC$, we obtain $\angle BXD = \angle DYC$. $\square$

3. We show that the only solution is $(p, q) = (13, 31)$.

First suppose that $p = q$. Then $p^2 - p - 1 = 2q + 3 = 2p + 3$, therefore $(p - 4)(p + 1) = 0$. As neither 4 and -1 are prime numbers, there are no solutions (p, q) with $p = q$.

Hence $p \neq q$, so from the equation, it follows that $p \mid 2q+3$ and $q \mid p^2-p-1$. As $2q+3$ and p^2-p-1 are positive, it follows that $p \leq 2q+3$ and $q \leq p^2-p-1$. To find a better lower bound for p, we multiply these relations:

$$\begin{aligned} pq &\mid (2q+3)(p^2-p-1) \\ &\mid 2qp^2 - 2qp - 2q + 3(p^2-p-1) \\ &\mid 3(p^2-p-1) - 2q. \end{aligned}$$

Note that $3(p^2 - p - 1) - 2q \geq 3q - 2q = q > 0$. Therefore the relation above yields

$$\begin{aligned} pq &\leq 3(p^2-p-1) - 2q \\ &= 3p^2 - 3p - (2q+3) \\ &\leq 3p^2 - 3p - p \\ &= 3p^2 - 4p. \end{aligned}$$

Adding $4p$ to both sides, then dividing both sides by p, we find that $q + 4 \leq 3p$, so

$$\frac{2q+3}{6} < \frac{q+4}{3} \leq p.$$

As p is a divisor of $2q+3$, we deduce that $2q+3 = kp$ with $k \in \{1, 2, 3, 4, 5\}$.

- If $k = 1$, then $2q + 3 = p$ and therefore also $q = p^2 - p - 1$. This implies that $p = 2q + 3 = 2(p^2 - p - 1) + 3 = 2p^2 - 2p + 1$. This in turn implies that $(2p - 1)(p - 1) = 0$, but this equation does not have prime solutions.
- Note that $k = 2$ and $k = 4$ are not possible either, since then kp would be even, while $2q + 3$ is odd.
- If $k = 3$, then it follows from $2q + 3 = 3p$ that $3 \mid q$, and therefore that $q = 3$. Hence $p = \frac{2q+3}{3} = 3$. However, this does not give a solution of the given equation.

Therefore $k = 5$. Then we have $5p = 2q + 3$, and therefore $5q = p^2 - p - 1$ as well. Substituting this gives $25p = 5(2q + 3) = 2(p^2 - p - 1) + 15 = 2p^2 - 2p + 13$, and therefore $(p - 13)(2p - 1) = 0$. As p is prime, it follows that $p = 13$, and that $q = \frac{5p-3}{2} = 31$. Note that $(p, q) = (13, 31)$ is indeed a solution of the given equation, so it is the only solution of the given equation. $\square$

4. We apply the AM-GM inequality on $\frac{1}{2}n(n-1)$ terms:

$$\frac{(n-1)\left(\frac{a_1}{a_2}\right)^{n-1} + (n-2)\left(\frac{a_2}{a_3}\right)^{n-1} + \ldots + \left(\frac{a_{n-1}}{a_n}\right)^{n-1}}{\frac{1}{2}n(n-1)}$$

$$\geq \left(\left(\frac{a_1}{a_2}\right)^{(n-1)^2}\left(\frac{a_2}{a_3}\right)^{(n-1)(n-2)}\cdots\left(\frac{a_{n-1}}{a_n}\right)^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$= \left(\left(\frac{a_1}{a_2}\right)^{n-1}\left(\frac{a_2}{a_3}\right)^{n-2}\cdots\left(\frac{a_{n-1}}{a_n}\right)\right)^{\frac{2}{n}}$$

$$= \left(a_1^{n-1}a_2^{-1}a_3^{-1}\cdots a_n^{-1}\right)^{\frac{2}{n}}$$

$$= a_1^2\left(a_1^{-1}a_2^{-1}a_2^{-1}\cdots a_n^{-1}\right)^{\frac{2}{n}}.$$

As it is given that $a_1a_2\cdots a_n = 1$, this equals a_1^2. Cyclically permuting the indices in this equation gives similar inequalities for each a_i^2. Summing these inequalities gives the required inequality: the right hand side is obviously $a_1^2 + a_2^2 + \ldots + a_n^2$, and on the left hand side we only have terms of the form $\left(\frac{a_i}{a_{i+1}}\right)^{n-1}$, each with a factor $\frac{1}{\frac{1}{2}n(n-1)}((n-1) + (n-2) + \ldots + 1) = 1$.

Equality holds if and only if equality holds for each of the n inequalities. For each of these, equality holds if and only if the $n-1$ terms are equal. For $n = 2$, therefore equality trivially holds, so equality holds for all a_1, a_2 with $a_1a_2 = 1$. For $n > 2$, equality holds if all cyclic permutations of $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_{n-1}}{a_n}$ hold. Combining two of these equations, we get $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_n}{a_1} = a$ for some positive real a, and if the latter equation holds, we see that all cyclic permutations of the former equation also hold.

By taking the product of all the fractions in the latter equation, we get $a^n = \frac{a_1}{a_2}\frac{a_2}{a_3}\cdots\frac{a_n}{a_1} = 1$, hence that $a = 1$. All a_i must therefore be equal, so by the given condition, they must all be equal to 1, and indeed, substituting $a_i = 1$ for all i in the inequality of the problem does give equality. $\square$

5. The largest possible value of k is $2^{10} - 10$. First note that there are 2^{10} possible combinations for the origins per kind of fish. We show that there are always at least 10 exceptions (combinations that cannot be obtained by a customer), and that there is a way to supply the stalls for which there are exactly 10 exceptions.

Let us number both the stalls and the kinds of fish from 1 up to 10. For stall i, define the sequence $a_i \in \{M, N\}^{10}$ as the sequence of origins of the 10 kinds of fish in this stall. Let c_i be the complement of a_i, i.e. the sequence obtained from a_i by replacing all M's with N's and vice versa. As every customer has bought a fish from stall i, no customer can have c_i as his sequence of fish origins. If all c_i (for $1 \leq i \leq 10$) are distinct, then we have 10 exceptions.

Otherwise, two of the stalls, say i and j, sell each kind of fish from the same origin, i.e. $a_i = a_j$, and therefore also $c_i = c_j$. Define the sequences d_k with $1 \leq k \leq 10$ by changing in c_i the origin of kind k of fish. These are the sequences which have exactly one origin in common with a_i. If a customer would have had sequence d_k, then this customer therefore could only have bought a fish from one of stalls i or j, but not both, contradicting the given that every customer bought exactly one fish from every stall. Therefore also in this case, there are at least 10 exceptions.

We now construct a market in which it is possible to buy $2^{10} - 10$ possible combinations of fish origins as in the problem. Suppose that stall i sells fish from the North Sea, unless the fish is of kind i (in which case the fish is from the Mediterranean Sea). Let $b \in \{M, N\}^{10}$ a sequence of origins in which the number of N's is not exactly 1. We show that we can buy 10 fish from 10 stalls in such a way that b is the sequence of origins. Let A be the set of indices i for which $b_i = M$, and B be the set of indices i for which $b_i = N$. For $i \in A$, buy a fish of kind i from stall i, so that we get a fish from the Mediterranean Sea. If B is empty, then we are done. If not, then B has at least two elements. Write $B = \{i_1, \ldots, i_n\} \subset \{1, \ldots, 10\}$. For $i_k \in B$, buy fish of kind i_k from stall i_{k+1}, considering the indices modulo n. Since $n \geq 1$, we have $i_{k+1} \neq i_k$, so this fish is from the North Sea, as required. $\square$

IMO Team Selection Test 1, June 2022

Problems

1. Determine all positive integers $n \geq 2$ which have a positive divisor $m \mid n$ satisfying

$$n = d^3 + m^3,$$

where d is the smallest divisor of n which is greater than 1.

2. Two circles Γ_1 and Γ_2 are given with centres O_1 and O_2 and common exterior tangents ℓ_1 and ℓ_2. The line ℓ_1 intersects Γ_1 in A and Γ_2 in B. Let X be a point on segment O_1O_2, not lying on Γ_1 or Γ_2. The segment AX intersects Γ_1 in $Y \neq A$ and the segment BX intersects Γ_2 in $Z \neq B$. Prove that the line through Y tangent to Γ_1 and the line through Z tangent to Γ_2 intersect each other on ℓ_2.

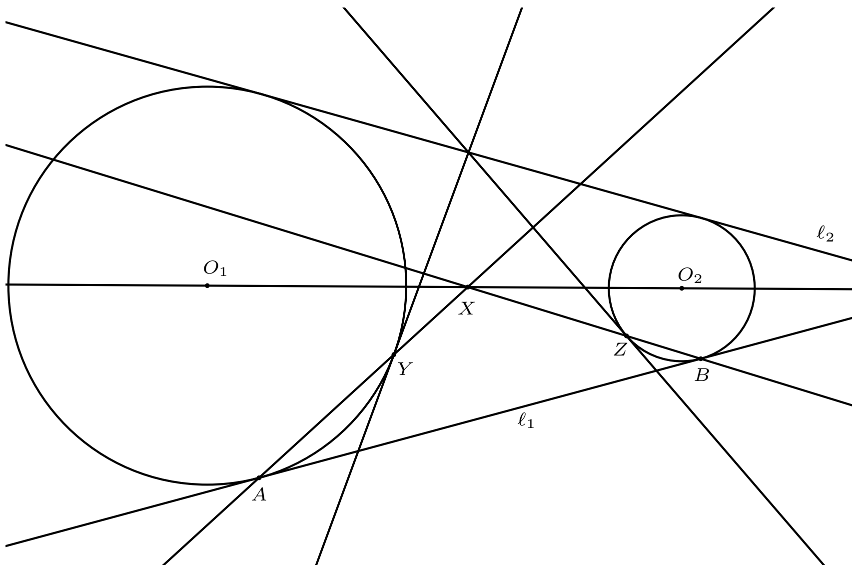
3. For real numbers x and y we define $M(x, y)$ to be the maximum of the three numbers xy, $(x-1)(y-1)$, and $x + y - 2xy$. Determine the smallest possible value of $M(x, y)$ where x and y range over all real numbers satisfying $0 \leq x, y \leq 1$.

4. In a sequence $a_1, a_2, \ldots, a_{1000}$ consisting of 1000 distinct numbers a pair (a_i, a_j) with $i < j$ is called *ascending* if $a_i < a_j$ and *descending* if $a_i > a_j$. Determine the largest positive integer k with the property that every sequence of 1000 distinct numbers has at least k non-overlapping ascending pairs or at least k non-overlapping descending pairs.

Solutions

1. The smallest divisor of n greater than 1 is the smallest prime divisor of n, hence d is prime. Moreover, we have $d \mid n$, hence $d \mid d^3 + m^3$, and $d \mid m^3$. This yields that $m > 1$. On the other hand we have $m \mid n$, hence $m \mid d^3 + m^3$, and $m \mid d^3$. Because d is prime and $m > 1$, we also see that m equals d, d^2, or d^3.

In all cases the parity of m^3 is equal to that of d^3, and $n = d^3 + m^3$ is even. This means that the smallest divisor of n greater than 1 equals 2, i.e. $d = 2$. In case $m = d$, we find $n = 2^3 + 2^3 = 16$, in case $m = d^2$, we find $n = 2^3 + 2^6 = 72$, and in case $m = d^3$, we find $n = 2^3 + 2^9 = 520$. These are indeed solutions: they are even so that $d = 2$, and $2 \mid 16$; $4 \mid 72$ and $8 \mid 520$, which indeed gives $m \mid n$. □



2. We consider the configuration in which Y lies between A and X; the other configurations are treated analogously. Let C be the intersection of ℓ_2 and Γ_1. Then C is the reflection of A in O_1O_2. We get

$$\begin{aligned}
\angle O_1YX &= 180^\circ - \angle O_1YA && \text{(straight angle)}\\
&= 180^\circ - \angle YAO_1 && (O_1YA \text{ is isosceles})\\
&= 180^\circ - \angle XAO_1 &&\\
&= 180^\circ - \angle XCO_1 && (A \text{ is the reflection of } C \text{ in } O_1X)
\end{aligned}$$

which yields that O_1CXY is cyclic.

Now let S be the intersection of the line through Y tangent to Γ_1, and the line ℓ_2. Then both SC and SY are tangent to Γ_1, hence we have $\angle SCO_1 = 90° = \angle SYO_1$, and O_1CSY is cyclic.

We see that both X and S lie on the circle through O_1, C, and Y. Therefore, we have $\angle SXO_1 = \angle SYO_1 = 90°$. We conclude that SX is perpendicular to O_1O_2. Analogously, for the intersection S' of the line through Z tangent to Γ_2, and the line ℓ_2, we can deduce that $S'X$ is perpendicular to O_1O_2. Because S and S' both lie on ℓ_2, we have $S = S'$. Hence the two tangents intersect each other on ℓ_2. $\square$

3. We will show that the minimum value is $\frac{4}{9}$. This value can be attained by taking $x = y = \frac{2}{3}$. Then we have $xy = \frac{4}{9}$, $(x-1)(y-1) = \frac{1}{9}$, and $x + y - 2xy = \frac{4}{9}$, and the maximum is indeed $\frac{4}{9}$.

Now we will prove that $M(x, y) \geq \frac{4}{9}$ for all x and y. Let $a = xy$, $b = (x-1)(y-1)$, and $c = x + y - 2xy$. If we replace x and y by $1-x$ and $1-y$, then a and b will be interchanged and c stays the same, because $(1-x)+(1-y)-2(1-x)(1-y) = 2-x-y-2+2x+2y-2xy = x+y-2xy$. Hence, $M(1-x, 1-y) = M(x, y)$. We have $x + y = 2 - (1-x) - (1-y)$, so at least one of $x + y$ and $(1-x) + (1-y)$ is greater than or equal to 1, which means that we may assume without loss of generality that $x + y \geq 1$.

Now write $x + y = 1 + t$ with $t \geq 0$. We also have $t \leq 1$, because $x, y \leq 1$ and hence $x + y \leq 2$. The inequality between the arithmetic and geometric mean yields

$$xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{(1+t)^2}{4} = \frac{t^2+2t+1}{4}.$$

We have $b = xy - x - y + 1 = xy - (1+t) + 1 = xy - t = a - t$, hence $b \leq a$. Moreover,

$$c = x+y-2xy \geq (1+t) - 2 \cdot \frac{t^2+2t+1}{4} = \frac{2+2t}{2} - \frac{t^2+2t+1}{2} = \frac{1-t^2}{2}.$$

If $t \leq \frac{1}{3}$, then we have $c \geq \frac{1-t^2}{2} \geq \frac{1-\frac{1}{9}}{2} = \frac{4}{9}$ and hence $M(x, y) \geq \frac{4}{9}$ as well. The remaining case is $t > \frac{1}{3}$. We have $c = x+y-2xy = 1+t-2a > \frac{4}{3} - 2a$. Moreover, $M(x, y) \geq \max(a, \frac{4}{3} - 2a)$, hence

$$3M(x, y) \geq a + a + (\tfrac{4}{3} - 2a) = \tfrac{4}{3},$$

which yields that $M(x, y) \geq \frac{4}{9}$.

We conclude that the minimum value of $M(x, y)$ is $\frac{4}{9}$. $\square$

4. We will prove that the greatest k is 333. First consider the sequence $1000, 999, 998, \ldots, 669, 668, 1, 2, 3, \ldots, 666, 667$. The first 333 numbers in the sequence are not usable in an ascending pair, because for each of these numbers the numbers left of it are all greater and the numbers right of it are all smaller. Therefore, for the ascending pairs only the last 667 numbers are available and that gives at most 333 non-overlapping ascending pairs. For a descending pair (a_i, a_j) with $i < j$ we get that a_i cannot be one of the numbers 1 through 667, because for each of these numbers there are only greater numbers right of it. Hence, a_i must be one of the first 333 numbers, from which we deduce that there can be at most 333 non-overlapping descending pairs. We conclude that no $k > 333$ will satisfy the conditions.

Now we will prove that for all $t \geq 1$, there are at least t non-overlapping ascending or t non-overlapping descending pairs in any sequence of $3t - 1$ distinct numbers. We will prove this by induction on t. For $t = 1$, the sequence has length 2 and this pair of numbers is either descending or ascending, which means the statement is correct. Now let $r \geq 1$ and suppose the statement is true for $t = r$. We consider the case $t = r + 1$ and take any sequence $a_1, a_2, \ldots, a_{3r+2}$ of $3r + 2$ distinct numbers. If the sequence is completely ascending, we can make neighbouring pairs which are all ascending. These are $\lfloor \frac{3r+2}{2} \rfloor \geq \frac{2r+2}{2} = r + 1$ pairs. Analogously, if the sequence is fully descending, there are at least $r + 1$ descending pairs. If the sequence is not fully ascending and also not fully descending, there is a spot in the sequence where the sequence is first ascending and then descending or the other way around. In other words: there are numbers a_i, a_{i+1}, a_{i+2} in the sequence with $a_i < a_{i+1} > a_{i+2}$ or $a_i > a_{i+1} < a_{i+2}$. In both cases, these three numbers contain both an ascending and descending pair. Now apply the induction hypothesis to the sequence $a_1, a_2, \ldots, a_{i-1}, a_{i+3}, a_{i+4}, \ldots, a_{3r+2}$. This is a sequence with $3r + 2 - 3 = 3r - 1$ distinct numbers, so there are at least r non-overlapping ascending pairs or r non-overlapping descending pairs. In the former case, we can add the ascending pair from a_i, a_{i+1}, a_{i+2} to it, and in the latter case, we can add the descending pair to it. In this way, we obtain $r + 1$ non-overlapping ascending pairs or $r + 1$ non-overlapping descending pairs. This completes the induction.

Now substitute $t = 333$ in this result: in a sequence consisting of 998 distinct numbers, there are always at least 333 non-overlapping ascending pairs or at least 333 non-overlapping descending pairs. This is also true for a sequence consisting of 1000 numbers (just ignore the last two numbers). Hence, $k = 333$ satisfies the conditions and is the greatest such k. $\square$

IMO Team Selection Test 2, June 2022

Problems

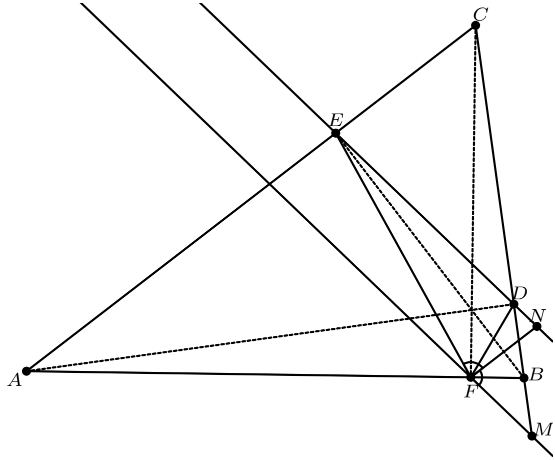
1. Consider an acute triangle ABC with $|AB| > |CA| > |BC|$. The vertices D, E, and F are the base points of the altitudes from A, B, and C, respectively. The line through F parallel to DE intersects BC in M. The angular bisector of $\angle MFE$ intersects DE in N. Prove that F is the circumcentre of $\triangle DMN$ if and only if B is the circumcentre of $\triangle FMN$.

2. Let n be a positive integer. For a real $x \geq 1$, assume that $\lfloor x^{n+1} \rfloor$, $\lfloor x^{n+2} \rfloor$, $\ldots$, $\lfloor x^{4n} \rfloor$ are all squares of positive integers. Prove that $\lfloor x \rfloor$ is also the square of a positive integer.
Here $\lfloor z \rfloor$ is the greatest integer smaller than or equal to z.

3. There are 15 lights on the ceiling of a room, numbered from 1 to 15. All lights are turned off. In another room, there are 15 switches: a switch for lights 1 and 2, a switch for lights 2 and 3, a switch for lights 3 en 4, et cetera, including a sqitch for lights 15 and 1. When the switch for such a pair of lights is turned, both of the lights change their state (from on to off, or vice versa). The switches are put in a random order and all look identical. Raymond wants to find out which switch belongs which pair of lights. From the room with the switches, he cannot see the lights. He can, however, flip a number of switches, and then go to the other room to see which lights are turned on. He can do this multiple times. What is the minimum number of visits to the other room that he has to take to determine for each switch with certainty which pair of lights it corresponds to?

4. Determine all positive integers d for which there exists a $k \geq 3$ such that you can put the numbers $d, 2d, 3d, \ldots, kd$ in a sequence in such a way that that the sum of every pair of neighbouring numbers is a square.

Solutions



1. Because of the requirement on the length, the configuration is fixed: M lies on the ray CB past B, and N lies on the ray ED past D. See the figure. Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Moreover, let H be the orthocentre of the triangle (in other words: the intersection of AD, BE, and CF). Thales's theorem yields that $AFHE$, $BDHF$, $CEHD$, $ABDE$, $BCEF$, and $CAFD$ are cyclic. Because of the cyclic quadrilateral $ABDE$, we get $\angle CED = 180° - \angle AED = \angle ABD = \beta$ and because of the cyclic quadrilateral $BCEF$, we get $\angle AEF = 180° - \angle CEF = \angle CBF = \beta$. Analogously, $\angle CDE$ and $\angle BDF$ equal α.

From $\angle CED = \beta = \angle AEF$ it follows that $\angle DEH = 90° - \beta = \angle FEH$. Hence, EH is the angular bisector of $\angle DEF$. Because $DE \parallel FM$, we get that $\angle MFE = 180° - \angle FED = 180° - 2(90° - \beta) = 2\beta$. As FN is the angular bisector of $\angle MFE$, we have $\angle EFN = \frac{1}{2} \cdot 2\beta = \beta$. Because $\angle FEH = 90° - \beta$, we also see that FN and EH are perpendicular, hence EH is not only the angular bisector in $\triangle FEN$, but it is also an altitude. Therefore, this line is also the perpendicular bisector of FN. As B lies on this line, we get $|BF| = |BN|$.

We already saw that $\angle CDE = \alpha = \angle BDF$. Because $DE \parallel FM$, we also have $\angle BMF = \angle CDE = \alpha$, hence $\angle DMF = \angle BMF = \angle BDF = \angle MDF$. Thus, $|FM| = |FD|$.

Let S be the intersection of AC with MF. Then we have $\angle BFM = \angle AFS$ and because $DE \parallel FM$, we get $\angle CED = \angle CSF$. The exterior angle theorem in triangle AFS yields that $\angle CSF = \angle SAF + \angle AFS = \alpha + \angle AFS$.

Combining everything, we obtain $\angle CED = \alpha + \angle BFM$. On the other hand, we knew that $\angle CED = \beta$, hence $\angle BFM = \beta - \alpha$. Moreover, we know that $\angle BMF = \alpha$. We conclude that $|BF| = |BM|$ if and only if $\beta - \alpha = \alpha$, or if and only if $\beta = 2\alpha$. Because we already know that $|BF| = |BN|$, we get: B is the circumcentre of $\triangle FMN$ if and only if $\beta = 2\alpha$.

Before, we saw that EH is the perpendicular bisector and altitude in triangle EFN, hence this triangle is isosceles with top angle E, which yields that $\angle DNF = \angle ENF = \angle EFN = \beta$. Moreover, we know that $\angle CDE = \alpha = \angle BDF$, from which it follows that $\angle NDF = \angle NDB + \angle BDF = \angle CDE + \angle BDF = 2\alpha$. Hence, $|FD| = |FN|$ if and only if $\beta = 2\alpha$. Because we already knew that $|FM| = |FD|$, we now get: F is the circumcentre of $\triangle DMN$ if and only if $\beta = 2\alpha$.

We conclude that F is the circumcentre of $\triangle DMN$ if and only if B is the circumcentre of $\triangle FMN$, as both properties are equivalent to $\beta = 2\alpha$. $\square$

2. We first prove the statement for $n = 1$. Write $x = a+r$, with $a \geq 1$ an integer and $0 \leq r < 1$. Suppose $\lfloor x^2 \rfloor$, $\lfloor x^3 \rfloor$ and $\lfloor x^4 \rfloor$ are squares. Then we have $a \leq x < a+1$, from which it follows that $a^2 \leq x^2 < (a+1)^2$. Hence, x^2 is squeezed between two consecutive squares. However, $\lfloor x^2 \rfloor$ is a square, hence the only possibility is that $\lfloor x^2 \rfloor = a^2$. We conclude that $a^2 \leq x^2 < a^2 + 1$. Completely analogously, we also get that $(a^2)^2 \leq x^4 < (a^2+1)^2$ and hence $\lfloor x^4 \rfloor = a^4$. We conclude that $a^4 \leq x^4 < a^4 + 1$.

Moreover, we have that $x^3 \geq a^3$. Now suppose that $x^3 \geq a^3 + 1$, i.e.

$$x^4 \geq x(a^3+1) = (a+r)(a^3+1) = a^4 + ra^3 + a + r \geq a^4 + a \geq a^4 + 1,$$

which gives a contradiction. Hence, $x^3 < a^3 + 1$, which yields that $\lfloor x^3 \rfloor = a^3$. This is also a square, hence a must be a square itself. We see that $\lfloor x \rfloor$ is a square.

Now we will finish the proof with induction to n. The induction basis has just been proved. Now let $k \geq 1$ and suppose that the statement is proved for $n = k$. Consider a real number $x \geq 1$ with the property that $\lfloor x^{k+2} \rfloor$, $\lfloor x^{k+3} \rfloor$, $\ldots$, $\lfloor x^{4k+4} \rfloor$ are all squares. In particular, $\lfloor x^{2(k+1)} \rfloor$, $\lfloor x^{3(k+1)} \rfloor$, and $\lfloor x^{4(k+1)} \rfloor$ are all squares. We can now apply the case $n = 1$ on x^{k+1} (which is a real number greater than or equal to 1) and find that $\lfloor x^{k+1} \rfloor$ is also a square. Now we know that $\lfloor x^{k+1} \rfloor$, $\lfloor x^{k+2} \rfloor$, $\ldots$, $\lfloor x^{4k} \rfloor$ are all squares and using the induction hypothesis, we obtain that $\lfloor x \rfloor$ is a square as well. This completes the proof by induction. $\square$

3. Walking back and forth just three times, Raymond cannot know all the switches with certainty. Indeed, if you note for each switch whether it is in the original or the switched state, there are $2^3 = 8$ different switching patterns. There are 15 switches, however, therefore there must be multiple switches with the same switching pattern and Raymond can never distinguish these switches from each other. Therefore, he has to go to the other room at least four times. We will prove that he can always do it in four times.

Starting from the situation in which all lights are turned off, suppose that Raymond changes some of the switches. If he changed none or all of the switches, then all lights are off. Otherwise, there is always a light which has been switched by exactly one switch, so this light is now on. Consider such a light i that is now turned on. Then the switch corresponding to $i-1$ and i or the one corresponding to i and $i+1$ has been flipped. (Consider the numbers of the lights modulo 15.) First suppose that it was the latter switch. Then consider light $i+1$. If this light is off, then the switch corresponding to $i+1$ and $i+2$ has also been flipped; if the light is on, then this switch has not been flipped. Next, we can use the state of light $i+2$ to deduce whether the switch corresponding to $i+2$ and $i+3$ has been flipped or not. Continuing this way, we can determine for each switch whether or not it has been flipped. In the second case, when the switch corresponding to $i-1$ and i has been flipped, it is also determined for every other siwtch whether or not it has been flipped. Hence, there are exactly two combinations of switches that give the same state for the lights. If we choose one of these combinations and then flip all 15 switches, exactly the same lights are turned on, hence this must be the second combination of switches. If the first combination of switches contains an even number of switched, the second combination contains 15 minus that number, an odd number.

Of course, Raymond knows how many switches he flipped. So using the state of the lights, he can deduce which of the two switch combinations is compatible with the total number of switches he flipped. Therefore, he can deduce exactly which switches have been flipped, but he does not know which switch is which in this combination of flipped switches.

Now write the numbers 1 to 15 in binary. Only 4 digits are needed for this; supplement the numbers by leading zeros so that each number has exactly 4 digits. Raymond numbers the switches with these binary numbers. Then in the first round, he flips the switches for which the first binary digit is a 1 and he writes down the 8 corresponding pairs of lights. In the second round, he first flips all switches back and flips the switches whose second binary digit is a 1. In the same way, for the third round he looks at the third binary digit, and in the fourth round at the fourth binary digit. In each round, he can figure out the pairs of lights for which the switch has been flipped (but not which switch is which). Because each switch corresponds to a unique subset of rounds, he can now find out for each switch which pair of lights corresponds to it. For example, if for some pair of lights the switch has been flipped in the first round, third round, fourth round, but not in the second round, then this corresponds to the switch with the binary code 1011, so with switch 11. Therefore, he can finish his task walking just four times to the other room. $\square$

4. For $d = 1$, we take $k = 15$ and the sequence

$$8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9.$$

Two neighbouring numbers in this sequence always add up to 9, 16, or 25. For square $d > 1$, we also take $k = 15$ and the same sequence as above, except that we multiply all numbers by d. Two neighbouring numbers in this sequence always add up to $9d$, $16d$, or $25d$, which are all squares.

Now consider a non-square d. We will show that this will not satisfy the conditions. Suppose that there does exists a k and a sequence $a_1d, a_2d, \ldots, a_kd$, such that $\{a_1, a_2, \ldots, a_k\} = \{1, 2, \ldots, k\}$. Write $d = cm^2$, where m is a positive integer such that c is not divisible by a square greater than 1. Then for all i with $1 \le i \le k-1$, we have that $a_id + a_{i+1}d$ is a square and $d = cm^2 \mid a_id + a_{i+1}d$, hence $cd = c^2m^2 \mid a_id + a_{i+1}d$, which yields that $c \mid a_i + a_{i+1}$. From this, we obtain that $a_{i+1} \equiv -a_i \mod c$ and hence $a_{i+2} \equiv -a_{i+1} \equiv a_i \mod c$. Therefore, there are at most two distinct residue classes modulo c occurring among the a_i, namely the classes of a_1 and a_2. However, $\{a_1, a_2, \ldots, a_k\} = \{1, 2, \ldots, k\}$ and $k \ge 3$, and therefore we must have $c \le 2$. Because d is not a square, $c = 1$ is impossible, hence $c = 2$. But then we have $a_{i+1} \equiv -a_i \equiv a_i \mod 2$, so there is at most one residue class modulo 2 occurring in the sequence, which gives a contradiction.

We conclude that the d that satisfy the conditions are the squares. $\square$

IMO Team Selection Test 3, June 2022

Problems

1. Find all quadruples (a, b, c, d) of non-negative integers such that $ab = 2(1 + cd)$ and there exists a non-degenerate triangle with sides of length $a - c$, $b - d$, and $c + d$.

2. Let $n > 1$ be an integer. There are n boxes in a row, and there are $n + 1$ identical stones. A *distribution* is a way to distribute the stones over the boxes, in which every stone is in exactly one of the boxes. We say that two of such distributions are *a stone's throw away* from each other if we can obtain one distribution from the other by moving exactly one stone from one box to another. The *cosiness* of a distribution a is defined as the number of distributions that are a stone's throw away from a. Determine the average cosiness of all possible distributions.

3. Find all natural numbers n for which there exists an integer $a > 2$ such that $a^d + 2^d \mid a^n - 2^n$ for all positive divisors $d \neq n$ of n.

4. Let $\triangle ABC$ be a triangle such that C is a right angle and $|AC| > |BC|$, let I be the centre of its incircle, and let H be the projection of C on the line segment AB. The incircle ω of $\triangle ABC$ is tangent to the sides BC, CA, and AB in the points A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in the lines A_1C_1 and B_1C_1, respectively, and let K and L be the reflections of H in the lines A_1C_1 and B_1C_1, respectively. Prove that the circumcircles of A_1EI, B_1FI, and C_1KL are concurrent.

Solutions

1. Note that $a > c$ and $b > d$, as $a - c$ and $b - d$ are sides of a non-degenerate triangle. So $a \geq c + 1$ and $b \geq d + 1$, as they are integers. Consider two cases: $a > 2c$ and $a \leq 2c$.

Suppose that $a > 2c$. Then $ab > 2bc \geq 2c \cdot (d + 1) = 2cd + 2c$. We also have $ab = 2 + 2cd$, so $2c < 2$, and therefore $c = 0$. We deduce that $ab = 2$ and that there exists a non-degenerate triangle with sides a, $b - d$, and d. Therefore $d \geq 1$ and $b > d$, so $b \geq 2$. From $ab = 2$ it follows that $a = 1$ and $b = 2$, and therefore also $d = 1$. Note that there exists a non-degenerate triangle with sides 1, 1, and 1, so the quadruple $(1, 2, 0, 1)$ is a solution.

Now suppose that $a \leq 2c$. By the triangle inequality, we have $(a - c) + (b - d) > c + d$, so $a + b > 2(c + d)$. As $a \leq 2c$, it follows that $b > 2d$. As $a \geq c+1$, we have $ab > (c+1) \cdot 2d = 2cd + 2d$. On the other hand, we have $ab = 2 + 2cd$, so $2d < 2$, and therefore $d = 0$. Analogously to the previous case, we deduce that the only other solution is $(2, 1, 1, 0)$.

Therefore the only solutions are the quadruples $(1, 2, 0, 1)$ and $(2, 1, 1, 0)$. $\square$

2. We call two distributions *neighbours* if they are a stone's throw away from each other.

We count the number N_k of distributions containing exactly k empty boxes, where $0 \leq k \leq n-1$ (since not all boxes can be empty). There are $\binom{n}{k}$ ways to choose the empty boxes and $\binom{(k+1)+(n-k)-1}{k+1} = \binom{n}{k+1}$ ways to fill the remaining $n-k$ boxes with $n+1 = (n-k)+(k+1)$ stones such that each of these boxes contains at least one stone (number of multisubsets of size $k+1$ from a set of $n-k$ boxes), yielding a total of $N_k = \binom{n}{k} \cdot \binom{n}{k+1}$ distributions containing exactly k empty boxes. Given such a distribution, we can move a stone from any of the $n - k$ non-empty boxes to any of the other $n - 1$ boxes, giving rise to $(n - k)(n - 1)$ distinct neighbouring distributions. So, the cosiness of each of these N_k distributions is $(n - k)(n - 1)$.

For $k' = (n-1)-k$, the number of distributions containing exactly k' empty boxes equals $N_{k'} = \binom{n}{k'} \cdot \binom{n}{k'+1} = \binom{n}{n-1-k} \cdot \binom{n}{n-k} = \binom{n}{k+1} \cdot \binom{n}{k}$, so $N_{k'} = N_k$. The cosiness of each of these N_k distributions is $(n-k')(n-1) = (k+1)(n-1)$. This means that the average cosiness for these two (possibly coinciding) sets of distributions is $\frac{(n-k)+(k+1)}{2}(n - 1) = \frac{n+1}{2}(n - 1)$. Being constant in k, this is also the overall average. $\square$

3. We show that n satisfies the given condition if and only if n is prime or n is a power of 2 (including $n = 1$).

If n is an odd prime, then the only proper divisor d of n is $d = 1$. Let $a = 2^k - 2$ with $3 \leq k \leq n+1$, e.g. $a = 6$. Then we need to check that $2^k - a + 2 = 2^k$ is a divisor of $(2^k - 2)^n - 2^n$. As this is the difference of two terms which contain exactly n factors 2, the difference contains $n+1$ factors 2.

If n is a power of 2, say $n = 2^m$ with $m \geq 0$. If $m = 0$, then there are no proper divisors of n, so n satisfies the given condition because it is an empty condition. If $m \geq 1$, then for all proper divisors d of n, the integer $e = \frac{n}{d}$ is even. Note that

$$a^n - 2^n = a^{de} - 2^{de} \equiv (-2^d)^e - 2^{de} = 2^n((-1)^e - 1) \mod a^d + 2^d$$

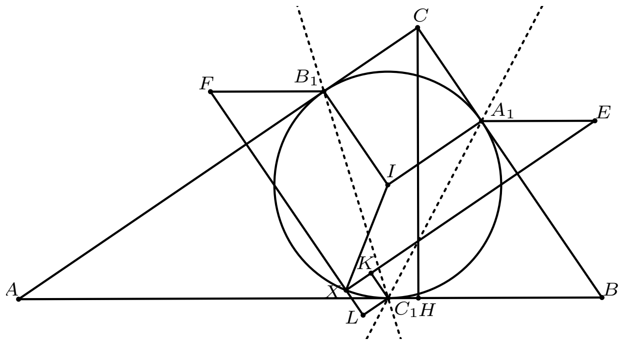
is zero for all a. Hence $a^n - 2^n$ is a multiple of $a^d + 2^d$, and therefore $a^d + 2^d \mid a^n - 2^n$. So if n is prime or a power of 2, n does indeed satisfy the given condition.

Finally, suppose that n is neither a prime number, nor a power of 2. Then we can write $n = de$ with $e \neq 1$ odd (since n is not a power of 2) and $d \neq 1$ (since n is not a prime number). As $(-1)^e - 1 = -2$, it follows from the computation above that $a^d + 2^d \mid 2^{n+1}$. Hence $a^d + 2^d$ is a power of 2, so $a^d = 2^k - 2^d$ for some $d < k \leq n+1$. Therefore a is divisible by 2 and $(\frac{a}{2})^d = 2^{k-d} - 1$.

Now we distinguish between the cases in which d is even and in which d is odd. In the first case $2^{k-d} - 1$ is a square. As $a > 2$, from $(\frac{a}{2})^d = 2^{k-d} - 1$ it however follows that $k - d \geq 2$, so this square is -1 modulo 4, which is a contradiction. If d is odd, then we note that

$$2^{k-d} = (\tfrac{a}{2})^d + 1 = (\tfrac{a}{2} + 1)((\tfrac{a}{2})^{d-1} - (\tfrac{a}{2})^{d-2} + \cdots + 1).$$

As $d \neq 1$, we have $\frac{a}{2} + 1 < (\frac{a}{2})^d + 1$. The second factor in the product above is a sum of an even number of terms with the same parity as $\frac{a}{2}$ and a term 1, so this factor is odd. As this second factor is also greater than 1, this contradicts it being a factor of a power of 2. $\square$



4. Note that the segments BC_1 and BA_1 have equal lengths because they both are tangents to ω through B. Therefore $\triangle A_1BC_1$ is an isosceles triangle with apex B, so the angular bisector of $\angle ABC$ is the altitude from B onto A_1C_1 and therefore perpendicular to A_1C_1.

We show that $A_1E \parallel AB$. As E is the reflection of C in A_1C_1, we have $\angle EA_1C_1 = \angle CA_1C_1$. As $\triangle A_1BC_1$ is isosceles with apex B, we have $\angle CA_1C_1 = \angle AC_1A_1$. So as $\angle EA_1C_1$ and $\angle AC_1A_1$ are alternate internal angles, we deduce that $A_1E \parallel AB$.

In the same way, we see that $C_1K \parallel BC$, as $\angle KC_1A_1 = \angle HC_1A_1 = \angle BC_1A_1 = \angle BA_1C_1$. Moreover, we have $|A_1E| = |A_1C| = r$ where r is the radius of the incircle ω, as C is a right angle.

We now show that $EK \perp BC$. Indeed, if S is the intersection of EK and BC, we see that $\angle A_1ES = \angle BCH$ as EK is the reflection of CH in A_1C_1. Moreover, $\angle EA_1S = \angle CBH$ as A_1E and BC are parallel (alternate internal angles). We deduce that $\triangle A_1ES$ and $\triangle BCH$ are similar, and that the angles $\angle ESA_1$ and $\angle CHB$ are right angles. As therefore EK and AC are both perpendicular on BC, it also follows that $EK \parallel AC$.

In the same way, we find $B_1F \parallel AB$, $C_1L \parallel AC$, $FL \parallel BC$, and $|B_1F| = r$.

Let X be the intersection of the lines EK and FL. We show that this is the point that lies on the circumcircles of A_1EI, B_1FI, and C_1KL.

First note that C_1KXL is a rectangle, and therefore a cyclic quadrilateral.

Now note that $|A_1E| = |B_1F| = r$. We show that $|XI| = r$. Let Y and Z be the intersections of the line through I parallel to AB and the lines EK and FL, respectively. Then A_1EYI and B_1FZI are parallellograms, so $|YI| = r = |ZI|$. Moreover $\angle YXZ = 90°$, so by Thales's theorem we also have $|XI| = r$. In particular, X lies on the incircle ω.

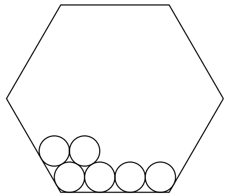
Now note that A_1EXI and B_1FXI are isosceles trapezoids, and therefore cyclic quadrilaterals. □

Junior Mathematical Olympiad, September 2019

Problems

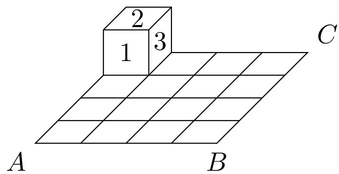
Part 1

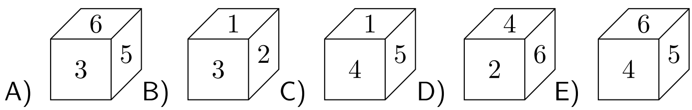
1. A regular hexagon is filled with small circles of the same size, as illustrated in the figure. The circles can be tangent, but they do not overlap. Exactly four circles fit next to each other along the side of the hexagon.
What is the maximum number of circles that fit in the hexagon in this way?



A) 30 B) 37 C) 39 D) 41 E) 44

2. The dice in the figure on the right has a 6 on the back (opposite the 1), a 5 on the bottom (opposite the 2), and a 4 on the left side (opposite the 3). The dice is tilted along the side of a 4×4 grid, on the little squares, until it lies on the little square at point B. This can be done along the side of the grid via A, or along the side of the grid via C.
How can the dice be positioned once it has arrived at B?





3. Ahmed, Babeth, Casper, Daan, Emine, and Freek are sitting in a row, in this order. Ahmed and Babeth both write a positive integer on a piece of paper. Then Casper adds the numbers on the papers of Ahmed and Babeth and writes the result on his piece of paper. Afterwards, Daan adds the numbers on the papers of Babeth and Casper and writes the result on his piece of paper. Then Emine adds the numbers on the papers of Casper and Daan and writes the result on her piece of paper. Finally, Freek adds the numbers on the papers of Daan and Emine and writes the result on his

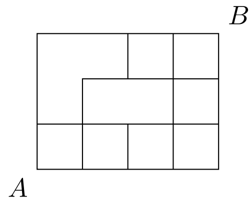
piece of paper.
Suppose that Emine wrote the number 19 on her paper, which number do you get if you add up the numbers on all papers?

A) 57 B) 76 C) 81 D) 89 E) 96

4. Farida makes a list of the integers between 1 and 10.000 that are divisible by 7. For every number on the list she adds the digits of the number. What is the smallest number that occurs as an outcome?

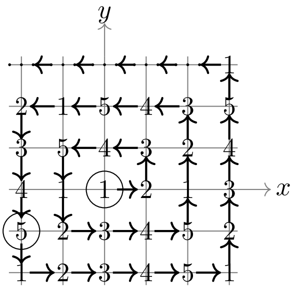
A) 1 B) 2 C) 3 D) 4 E) 5

5. An ant walks over the lines in the figure on the right and takes a shortest route from A to B.
How many routes are possible for the ant?



A) 7 B) 12 C) 18 D) 20 E) 30

6. In a grid the grid points are visited in a spiral, in counter clockwise direction. On every grid point a number from the list $1, 2, 3, 4, 5$ is written by starting with 1 at the point $(0, 0)$ and repeating the list indefinitely, like in the figure. The circled 1 is at the point $(0, 0)$ and the circled 5 is at the grid point $(-2, -1)$.
Which number is written at the grid point $(-20, 21)$?

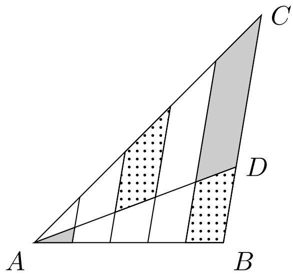


A) 1 B) 2 C) 3 D) 4 E) 5

7. A watchmaker installed the big and small hand of a clock in the wrong way. This makes the small hand go with the speed of the big hand and the big hand with the speed of the small hand. It is known that every day at 8:00 the clock shows the right time.
How many times a day (24 hours) does the clock show the right time?

A) 1 B) 2 C) 6 D) 22 E) 24

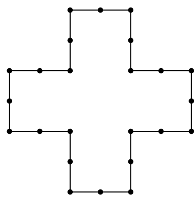
8. Given is the triangle ABC. A line from point A intersects the side BC in D. Parallel to BC we draw four lines such that they divide AB and AC in five equal parts. From the ten pieces in which the triangle ABC is divided, the two dotted ones have the same area. Also, the area of the grey triangle at the bottom left next to A is 5.
What is the area of the grey quadrilateral at the upper right next to C?



A) 50 B) 81 C) 100 D) 119 E) 121

Part 2

1. In the rectangular cross on the right all sides have the same length. The vertices and midpoints of the sides are marked with dots. A straight line segment is called a *halving segment* if it passes through two of these dots, and it divides the cross into two parts of equal area.
How many halving segments does the cross have?



2. An artist has an extraordinary working rhythm. He works for 3 hours very intensively on his art, and then he sleeps for 8 hours before starting to work again. Suppose that he starts working at midnight in the night from 31 July to 1 August.
Which day of August is the first day after 1 August on which the artist is working the same number of hours as on 1 August?

3. On a school there are between 500 and 1000 students. The gymnastics teacher wants to divide the students into teams of eight persons each for a sports day. Three students are left over. If the teacher tries to divide the students into teams of nine students, again three students will be left over. Also with teams of ten students each, three students will be left over.
How many students attend the school?

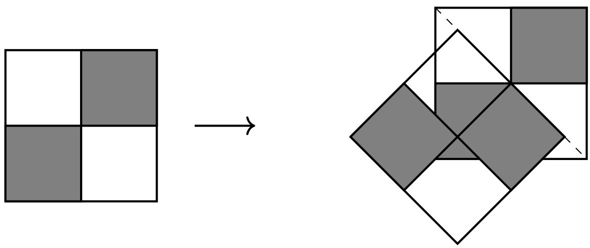
4. Liselotte has a collection of 100 candies, which are either sweet or bitter. She wants to choose between the following possibilities.

I) She eats half of the sweet candies. The rest is kept in the bag.

II) She eats half of the bitter candies. The rest is kept in the bag.

The part of the remaining candies in case I that are bitter, is three times as large as the part of remaining candies in case II that are bitter.
How many bitter candies does the bag contain (before Liselotte eats any of them)?

5. A square with area 4 is divided into two grey and two transparent squares, each having an area of size 1; see the left figure. Another such square is put on top of this square. The side of the second square is lying exactly on the middle of the diagonal of the first square; see the right figure.



What is the area of the grey part in the right figure? Give your answer as a reduced fraction.

6. In a cafe, each product costs at most 12 ducats. Currently the cafe owner is only using coins worth one ducat. This is unpractical for the more expensive products, however. Therefore, the cafe owner has decided to introduce two types of coins next to the coins of one ducat. He is doing this in such a way that as many values from 1 to 12 ducats can be paid with at most two coins (without change).
What is the worth of the two new types of coins?

7. A four digit number $\overline{aabb}$, that is, the number whose digits are a, a, b, and b, is the square of an integer.
Of which integer is $\overline{aabb}$ the square?

8. We compute the product of two numbers,

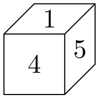
$$99\ldots99 \times 99\ldots99,$$

where the first number consists of 20 nines, and the second of 21 nines.
Which number do you get if you add up the digits of the outcome of this multiplication?

Answers

Part 1

1. B) 37

2. C) 

3. B) 76

4. B) 2

5. D) 20

6. D) 4

7. D) 22

8. B) 81

Part 2

1. 12

2. 7 August

3. 723

4. 20

5. $3\frac{1}{2}$

6. 4 and 6

7. 88

8. 189

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