59th Dutch Mathematical Olympiad 2020

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2021 started with the first round in January 2020, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 7928 students from 328 secondary schools participated.

The 944 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 128 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 59th edition 2020.

The 30 most outstanding candidates of the Dutch Mathematical Olympiad 2020 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO in Georgia, from 9 until 15 April. At this virutal event the Dutch team won one bronze medal. For more information about the EGMO (including the 2021 paper), see www.egmo.org.

In March a selection test of 3.5 hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), also a virtual event held on 1 and 2 May. The Dutch team achieved an outstanding result: three gold medals, two silver medals and three bronze medals. For more information about the BxMO (including the 2021 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2021 was selected by three team selection tests on 2, 3 and 4 June, each lasting 4 hours. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Egmond aan Zee from 10 until 18 July.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for the virtual IMO 2021 consists of

- Jelle Bloemendaal (17 years old)
  - bronze medal at BxMO 2019, silver medal at BxMO 2020 and 2021
  - (virtual) observer C at IMO 2020
- Kevin van Dijk (17 years old)
  - bronze medal at BxMO 2020, gold medal at BxMO 2021
  - (virtual) observer C at IMO 2020
- Hylke Hoogeveen (16 years old)
  - bronze medal at BxMO 2020, honourable mention at BxMO 2021
- Casper Madlener (16 years old)
  - silver medal at BxMO 2020
  - (virtual) observer C at IMO 2020
- Kees den Tex (17 years old)
  - gold medal at BxMO 2021
- Thian Tromp (18 years old)
  - bronze medal at BxMO 2020, silver medal at BxMO 2021

Also part of the IMO selection, but not officially part of the IMO team, is:

- Lars Pos (17 years old)
  - bronze medal at BxMO 2021

The team is coached by

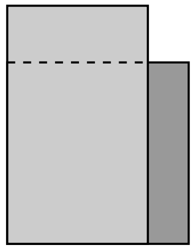
- Quintijn Puite (team leader), Eindhoven University of Technology
- Johan Konter (deputy leader), Stockholm University
- Ward van der Schoot (observer A), University of Cambridge

First Round, January 2020

Problems

A-problems

1. Francisca has a square piece of paper whose sides have length 10 cm. She also has a rectangular piece of paper having the exact same area as the square piece of paper. She puts the rectangle right on top of the square, putting the left bottom corner of both pieces of paper in the same spot. Exactly one quarter of the square remains uncovered by the rectangle. What is the length in centimetres of the long side of the rectangle?



A) 12 B) $12\frac{1}{4}$ C) $12\frac{1}{2}$ D) $12\frac{3}{4}$ E) $13\frac{1}{3}$

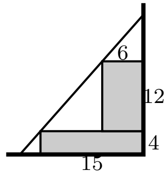
2. Each of Kwik, Kwek, and Kwak is lying on two consecutive days of the week and is telling the truth on the other five days. No two of them are lying on the same day. Uncle Donald wants to know who of his nephews ate his sweets. The three nephews know all too well who did it. On Sunday, Kwik says that Kwek ate the sweets. On Monday, Kwik says that it actually was not Kwek who ate the sweets, while Kwak claims that Kwik is innocent. On Tuesday, however, Kwak says that it was Kwik who ate the sweets. Who ate the sweets?

A) It was Kwik.
B) It was Kwek.
C) It was Kwak.
D) It was either Kwik or Kwek, but you cannot determine who of the two.
E) It was either Kwik or Kwak, but you cannot determine who of the two.

3. We consider numbers with two digits (the first digit cannot be 0). Such a number is called *vain* if the sum of the two digits is greater than or equal to the product of the two digits. For example, the number 36 is *not* vain, as $3 + 6$ is smaller than $3 \cdot 6$.
How many numbers with two digits are vain?

A) 17 B) 18 C) 26 D) 27 E) 37

4. A box measuring 4 dm by 15 dm is shoved against the wall. On top of it, a second box, measuring 12 dm by 6 dm, is placed. A ladder exactly touches the ground, the two boxes and the wall. See the figure (which is not drawn to scale).
What is the length of the ladder in dm?



A) 30 B) $8\sqrt{15}$ C) 31 D) $22\sqrt{2}$ E) $18\sqrt{3}$

5. On a 4×4 board, there are 16 grass hoppers, each on its own square. At a certain time, each grass hopper jumps to an adjacent square: to the square above, below, left, or right of its current square, but not diagonally and not leaving the board.
What is the maximum number of squares that can be empty after the grass hoppers have jumped?

A) 8 B) 9 C) 10 D) 11 E) 12

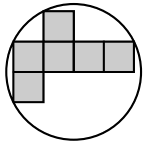
6. In the table below each of the three rows is a correct calculation (the symbol $\div$ denotes division). Also each of the three columns (read from top to bottom) is a correct calculation. However, the digits in the table have been replaced by letters. Different letters represent different digits and no digits are 0.

$$\begin{array}{ccccc} \text{ABC} & - & \text{ADF} & = & \text{F} \\ + & & - & & - \\ \text{ADD} & \div & \text{GC} & = & \text{C} \\ = & & = & & = \\ \text{CEF} & \div & \text{GD} & = & \text{D} \end{array}$$

Which digit does E represent?

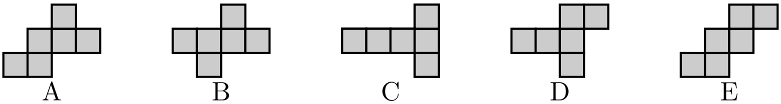
A) 1 B) 3 C) 5 D) 7 E) 9

7. We consider figures consisting of six squares whose sides have length 1. The *radius* of such a figure is the radius of the smallest circle containing the whole figure. On the right, there is an example of a figure with radius $\sqrt{5}$.

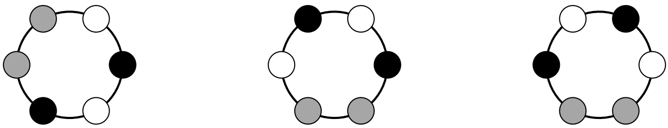


Which of the following five figures has the smallest radius?

A) A B) B C) C D) D E) E



8. Lieneke is making bracelets with beads. Each bracelet has six beads: two white, two grey, and two black beads. Some bracelets look different on first sight, but are actually not different: by turning or flipping the first one over, it looks the same as the other one. For example, the following three bracelets are the same.



How many really different bracelets can Lieneke make?

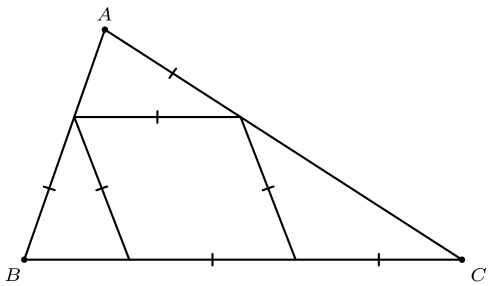
A) 10 B) 11 C) 12 D) 14 E) 15

B-problems

The answer to each B-problem is a number.

1. By replacing each $*$ in the expression $1 * 2 * 3 * 4 * 5 * \cdots * 2019 * 2020$ by a $+$ or a $-$ sign, we get a long calculation. Put the $+$ and $-$ signs in such a way that the outcome is a positive number (greater than 0) which is as small as possible.
What is this outcome?

2. Triangle ABC is subdivided into three isosceles triangles and a rhombus.
Note: the figure is not drawn to scale.



What is the size of angle C in degrees?

3. Annemiek and Bart each have a note on which they have written three different positive integers. It appears that there is exactly one number that is on both their notes. Moreover, if you add any two different numbers from Annemiek's note, you get one of the numbers on Bart's note. One of the numbers on Annemiek's note is her favourite number, and if you multiply it by 3, you get one of the numbers on Bart's note. Bart's note contains the number 25, his favourite number.
What is Annemiek's favourite number?

4. We consider rows of 2020 coins. Each coin is of denomination 1, 2, or 3. Between two coins of denomination 1, there is at least one other coin. Between two coins of denomination 2, there are at least two other coins. Between two coins of denomination 3, there are at least three other coins.
How many different rows of 2020 coins satisfy these conditions?

Solutions

A-problems

1. E) $13\frac{1}{3}$

2. C) It was Kwak.

3. D) 27

4. A) 30

5. C) 10

6. E) 9

7. B) B

8. B) 11

B-problems

1. 2

2. 36

3. 5

4. 10

Second Round, March 2020

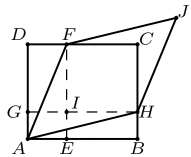
Problems

B-problems

The answer to each B-problem is a number.

B1. The *digit sum* of a number is obtained by adding all digits of the number. For example, the digit sum of 1303 is $1 + 3 + 0 + 3 = 7$.
Find the smallest positive integer n for which both the digit sum of n and the digit sum of $n + 1$ are divisible by 5.

B2. Rectangle $ABCD$ is subdivided into four rectangles as in the figure. The area of rectangle $AEIG$ is 3, the area of rectangle $EBHI$ is 5, and the area of rectangle $IHCF$ is 12.
What is the area of the parallelogram $AHJF$?



B3. A square sheet of paper lying on the table is divided into $8 \times 8 = 64$ equal squares. These squares are numbered from **a1** to **h8** as on a chess board (see fig. 1). We now start folding, in such a way that square **a1** always stays in the same spot on the table. First we fold along the horizontal midline (fig. 1). This will cause square **a8** to fold on top of square **a1**. Then we fold along the vertical midline (fig. 2). Next, we fold along the new horizontal midline (fig. 3), et cetera. After folding six times, we have a small package of paper in front of us (fig. 7) that we can consider as a stack of 64 square pieces of paper.

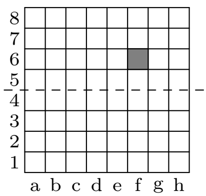


fig. 1

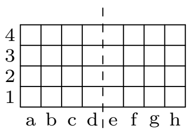


fig. 2

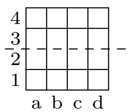


fig. 3

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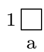


fig. 7

The squares in this stack are numbered from bottom to top from 1 to 64. So square **a1** gets number 1.
Which number does square **f6** get?

B4. One hundred brownies (girl scouts) are sitting in a big circle around the camp fire. Each brownie has one or more chestnuts and no two brownies have the same number of chestnuts. Each brownie divides her number of chestnuts by the number of chestnuts of her *right* neighbour and writes down the remainder on a green piece of paper. Each brownie also divides her number by the number of chestnuts of her *left* neighbour and writes down the remainder on a red piece of paper. For example, if Anja has 23 chestnuts and her right neighbour Bregje has 5, then Anja writes 3 on her green piece of paper and Bregje writes 5 on her red piece of paper.

If the number of distinct remainders on the 100 green pieces of paper equals 2, what is the smallest possible number of distinct remainders on the 100 red pieces of paper?

B5. Given is the sequence of numbers $a_0, a_1, a_2, \ldots, a_{2020}$ with $a_0 = 0$. Furthermore, the following holds for every $k = 1, 2, \ldots, 2020$:

$$a_k = \begin{cases} a_{k-1} \cdot k & \text{if } k \text{ is divisible by } 8, \\ a_{k-1} + k & \text{if } k \text{ is } \textit{not} \text{ divisible by } 8. \end{cases}$$

What are the last two digits of a_{2020}?

C-problems

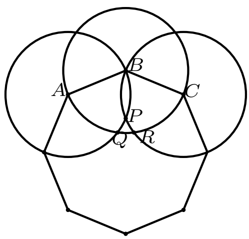
For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

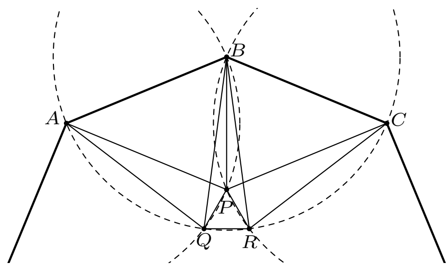
C1. Given a positive integer n, we denote by $n!$ ('n factorial') the number we get if we multiply all integers from 1 to n. For example: $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

(a) Determine all integers n with $1 \leq n \leq 100$ for which $n! \cdot (n+1)!$ is a perfect square. *Also, prove that you have found all solutions n.*

(b) Prove that no positive integer n exists such that $n! \cdot (n+1)! \cdot (n+2)! \cdot (n+3)!$ is a perfect square.

C2. Three consecutive vertices A, B, and C of a regular octagon (8-gon) are the centres of circles that pass through neighbouring vertices of the octagon. The intersection points P, Q, and R of the three circles form a triangle (see figure).





Prove that triangle PQR is equilateral.

Solutions

B-problems

1. 49999

2. $24\frac{1}{5}$

3. 43

4. 100

5. 02

C-problems

C1. (a) We observe that $(n+1)! = (n+1)\cdot n!$, and therefore that $n!\cdot(n+1)! = (n!)^2\cdot(n+1)$. That product is a perfect square if and only if $n+1$ is a perfect square, since $(n!)^2$ is a perfect square. For $1 \leq n \leq 100$ this is the case for $n = 3, 8, 15, 24, 35, 48, 63, 80, 99$ (perfect squares minus one that are below 100). □

(b) We rewrite the product $n!\cdot(n+1)!\cdot(n+2)!\cdot(n+3)!$ as follows:

$$(n!)^2\cdot(n+1)\cdot(n+2)!\cdot(n+3)! = (n!)^2\cdot(n+1)\cdot((n+2)!)^2\cdot(n+3).$$

Since $(n!)^2$ and $((n+2)!)^2$ are both perfect squares, the above product is a perfect square if and only if $(n+1)(n+3)$ is a perfect square. However, $(n+1)(n+3)$ cannot be a perfect square. Indeed, suppose that $(n+1)(n+3) = k^2$ were a perfect square. Since $(n+1)^2 < (n+1)(n+3) < (n+3)^2$ we would have $n+1 < k < n+3$, so $k = n+2$. This is impossible because $(n+1)(n+3) = (n+2)^2 - 1$, which is not equal to $(n+2)^2$. □

C2. An octagon can be subdivided into six triangles (see figure on the left). Together, the angles of those six triangles add up to the same number of degrees as the eight angles of the octagon. Since the angles of any triangle add up to 180 degrees, this means that the eight angles of the octagon add up to $6\cdot 180° = 1080°$. Hence, each of the angles of the regular octagon is $\frac{1}{8}\cdot 1080° = 135°$.

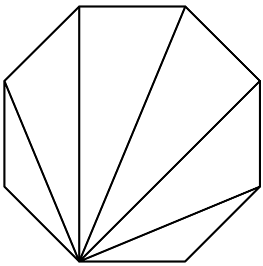
We now consider the figure from the problem statement (see figure on the right). Line segment BP bisects angle ABC, so $\angle ABP = \angle PBC = 67\frac{1}{2}^\circ$. Since triangles ABP and BCP are isosceles (as $|AB| = |AP|$ and $|BC| = |CP|$), we also have $\angle APB = \angle BPC = 67\frac{1}{2}^\circ$ and $\angle BAP = \angle BCP = 180^\circ - 135^\circ = 45^\circ$.

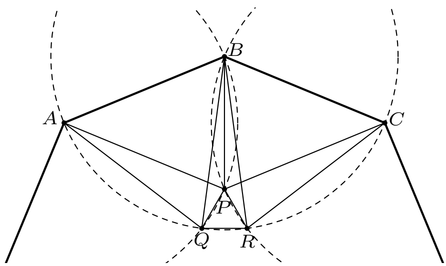
In triangles ABQ and BCR all sides have the same length. These triangles are therefore equilateral and all angles are 60°. From this, we deduce that $\angle PAQ = \angle BAQ - \angle BAP = 15^\circ$. In the same way, we find $\angle PCR = 15^\circ$. Furthermore, triangles PAQ and PCR are isosceles (since $|AP| = |AQ|$ and $|CP| = |CR|$), so $\angle APQ = \frac{1}{2}(180^\circ - 15^\circ) = 82\frac{1}{2}^\circ$ and $\angle CPR = 82\frac{1}{2}^\circ$.

By mirror symmetry, PQ and PR have the same length, so PQR is an isosceles triangle with apex P. We have already determined all angles at P, except $\angle QPR$. We deduce that

$$\begin{aligned}\angle QPR &= 360^\circ - \angle APQ - \angle APB - \angle BPC - \angle CPR \\ &= 360^\circ - 2 \cdot 67\tfrac{1}{2}^\circ - 2 \cdot 82\tfrac{1}{2}^\circ = 60^\circ.\end{aligned}$$

From this and the fact that PQR is isosceles, we directly conclude that PQR is equilateral. □





Final Round, September 2020

Problems

1. Daan distributes the numbers 1 to 9 over the nine squares of a 3×3-table (each square receives exactly one number). Then, in each row, Daan circles the median number (the number that is neither the smallest nor the largest of the three). For example, if the numbers 8, 1, and 2 are in one row, he circles the number 2. He does the same for each column and each of the two diagonals. If a number is already circled, he does not circle it again.

⑧	1	②
7	⑥	③
9	⑤	4

He calls the result of this process a *median table*. Above, you can see a median table that has 5 circled numbers.

(a) What is the **smallest** possible number of circled numbers in a median table?
Prove that a smaller number is not possible and give an example in which a minimum number of numbers is circled.

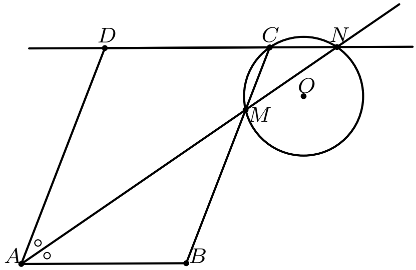
(b) What is the **largest** possible number of circled numbers in a median table?
Prove that a larger number is not possible and give an example in which a maximum number of numbers is circled.

2. For a given value t, we consider number sequences $a_1, a_2, a_3, \ldots$ such that $a_{n+1} = \dfrac{a_n + t}{a_n + 1}$ for all $n \geq 1$.

(a) Suppose that $t = 2$. Determine all starting values $a_1 > 0$ such that $\frac{4}{3} \leq a_n \leq \frac{3}{2}$ holds for all $n \geq 2$.

(b) Suppose that $t = -3$. Investigate whether $a_{2020} = a_1$ for all starting values a_1 different from -1 and 1.

3. Given is a parallelogram $ABCD$ with $\angle A < 90°$ and $|AB| < |BC|$. The angular bisector of angle A intersects side BC in M and intersects the extension of DC in N. Point O is the centre of the circle through M, C, and N.
Prove that $\angle OBC = \angle ODC$.



4. Determine all pairs of integers (x, y) such that $2xy$ is a perfect square and $x^2 + y^2$ is a prime number.

5. Sabine has a very large collection of shells. She decides to give part of her collection to her sister.

On the first day, she lines up all her shells. She takes the shells that are in a position that is a perfect square (the first, fourth, ninth, sixteenth, etc. shell), and gives them to her sister. On the second day, she lines up her remaining shells. Again, she takes the shells that are in a position that is a perfect square, and gives them to her sister. She repeats this process every day.

The 27th day is the first day that she ends up with fewer than 1000 shells. The 28th day she ends up with a number of shells that is a perfect square for the tenth time.
What are the possible numbers of shells that Sabine could have had in the very beginning?

Solutions

1. (a) The smallest possible number of circled numbers is 3. Fewer than 3 is not possible since in each row at least one number is circled (and these are three different numbers).

On the right, a median table is shown in which only 3 numbers are circled. In the rows, the numbers $7, 5, 3$ are circled, in the columns the numbers $3, 5, 7$, and on the diagonals the numbers 5 and 5. Together, these are three different numbers: 3, 5, and 7.

4	9	(7)
2	(5)	8
(3)	1	6

(b) The largest possible number of circled numbers is 7. More than 7 is not possible, since the numbers 9 and 1 are never circled, hence no more than $9 - 2 = 7$ numbers are circled.

On the right, a median table is shown in which 7 numbers are circled. In the rows, the numbers $2, 6, 8$ are circled, in the columns the numbers $7, 5, 3$, and on the diagonals the numbers 4 and 5. Together, these are the numbers $2, 3, 4, 5, 6, 7, 8$.

(4)	1	(2)
(7)	(5)	(6)
(8)	9	(3)

2. (a) First, we determine for what starting values $a_1 > 0$ the inequalities $\frac{4}{3} \leq a_2 \leq \frac{3}{2}$ hold. Then, we will prove that for those starting values, the inequalities $\frac{4}{3} \leq a_n \leq \frac{3}{2}$ are also valid for all $n \geq 2$.

First, we observe that $a_2 = \frac{a_1+2}{a_1+1}$ and that the denominator, $a_1 + 1$, is positive (since $a_1 > 0$). The inequality

$$\frac{4}{3} \leq a_2 = \frac{a_1 + 2}{a_1 + 1} \leq \frac{3}{2},$$

is therefore equivalent to the inequality

$$\tfrac{4}{3}(a_1 + 1) \leq a_1 + 2 \leq \tfrac{3}{2}(a_1 + 1),$$

as we can multiply all parts in the inequality by the positive number $a_1 + 1$. Subtracting $a_1 + 2$ from all parts of the inequality, we see that this is equivalent to

$$\tfrac{1}{3}a_1 - \tfrac{2}{3} \leq 0 \leq \tfrac{1}{2}a_1 - \tfrac{1}{2}.$$

We therefore need to have $\frac{1}{3}a_1 \leq \frac{2}{3}$ (i.e. $a_1 \leq 2$), and $\frac{1}{2} \leq \frac{1}{2}a_1$ (i.e. $1 \leq a_1$). The starting value a_1 must therefore satisfy $1 \leq a_1 \leq 2$.

Now suppose that $1 \leq a_1 \leq 2$, so that a_2 satisfies $\frac{4}{3} \leq a_2 \leq \frac{3}{2}$. Looking at a_3, we see that $a_3 = \frac{a_2+2}{a_2+1}$. That is the same expression as for a_2,

only with a_1 replaced by a_2. Since a_2 also satisfies $1 \leq a_2 \leq 2$, the same argument now shows that $\frac{4}{3} \leq a_3 \leq \frac{3}{2}$.
We can repeat the same argument to show this for a_4, a_5, etcetera. Hence, we find that $\frac{4}{3} \leq a_n \leq \frac{3}{2}$ holds for all $n \geq 2$. The formal proof is done using induction: the induction basis $n = 2$ has been shown above. For the induction step, see the solution of part (b) of the version for klas 5 & klas 4 and below. The result is that all inequalities hold if and only if $1 \leq a_1 \leq 2$.

(b) Let's start by computing the first few numbers of the sequence in terms of a_1. We see that

$$a_2 = \frac{a_1 - 3}{a_1 + 1}$$

and

$$\begin{aligned} a_3 &= \frac{a_2 - 3}{a_2 + 1} = \frac{\frac{a_1-3}{a_1+1} - 3}{\frac{a_1-3}{a_1+1} + 1} = \frac{a_1 - 3 - 3(a_1 + 1)}{a_1 - 3 + (a_1 + 1)} \\ &= \frac{-2a_1 - 6}{2a_1 - 2} = \frac{-a_1 - 3}{a_1 - 1}. \end{aligned}$$

Here, it is important that we do not divide by zero, that is, $a_1 \neq -1$ and $a_2 \neq -1$. The first inequality follows directly from the assumption. For the second inequality we consider when $a_2 = -1$ holds. This is the case if and only if $a_1 - 3 = -(a_1 + 1)$, if and only if $a_1 = 1$. Since we assumed that $a_1 \neq 1$, we see that $a_2 \neq -1$. The next number in the sequence is

$$a_4 = \frac{a_3 - 3}{a_3 + 1} = \frac{\frac{-a_1-3}{a_1-1} - 3}{\frac{-a_1-3}{a_1-1} + 1} = \frac{-a_1 - 3 - 3(a_1 - 1)}{-a_1 - 3 + (a_1 - 1)} = \frac{-4a_1}{-4} = a_1.$$

Again, we are not dividing by zero since $a_3 = -1$ only holds when $-a_1 - 3 = -a_1 + 1$, which is never the case.
We see that $a_4 = a_1$. Since a_{n+1} only depends on a_n, we see that $a_5 = a_2$, $a_6 = a_3$, $a_7 = a_4$, et cetera. In other words: the sequence is periodic with period 3, and we see that

$$a_{2020} = a_{2017} = a_{2014} = \cdots = a_4 = a_1.$$

To conclude: indeed we have $a_{2020} = a_1$ for all starting values a_1 unequal to 1 and -1.

3. As an intermediate step, we first show that triangles OCM and OCN are congruent. Since AD and BC are parallel, we have (F angles): $\angle CMN =$

$\angle DAM = \frac{1}{2}\angle DAB$. Since DN and AB are parallel, we have (Z angles): $\angle CNM = \angle NAB = \frac{1}{2}\angle DAB$. It follows that $\angle CMN = \angle CNM$, so triangle CMN is isosceles with apex C. We obtain $|CM| = |CN|$. Line segments OC, ON, and OM are radii of the same circle, and therefore of equal length. Triangles OCM and OCN are therefore congruent (three pairs of equal sides).

To show that $\angle OBC = \angle ODC$, we will show that triangles OBC and ODN are congruent. We will do this using the ZHZ-criterion. We will show that $\angle OND = \angle OCB$, and $|ON| = |OC|$, and $|DN| = |BC|$.

The equality $|ON| = |OC|$ follows since ON and OC are radii of the same circle. In part (a), we saw triangles OCM and OCN are congruent. Furthermore, these two triangles are isosceles ($|OC| = |OM|$ and $|OC| = |ON|$). Hence, the four base angles $\angle ONC$, $\angle OCN$, $\angle OMC$, and $\angle OCM$ are equal. We see that $\angle OND = \angle OCB$. The only thing we still need to show is that $|DN| = |BC|$.

Observe that $\angle BMA = \angle DAM$ (Z angles) and $\angle DAM = \angle MAB$ (as AM is the angular bisector of A). We find that $\angle BMA = \angle MAB$. Triangle AMB is therefore isosceles and we have $|AB| = |BM|$. We previously saw that $|CM| = |CN|$, and we also have $|AB| = |CD|$ as $ABCD$ is a parallelogram. We therefore obtain

$$|DN| = |CD| + |CN| = |AB| + |CM| = |BM| + |CM| = |BC|,$$

which concludes the proof.

4. We have $2xy = a^2$ for some nonnegative integer a, and $x^2 + y^2 = p$ for some prime number p.

Since a prime number is never a perfect square, we see that $x, y \neq 0$. Since $2xy$ is a perfect square, it follows that x and y must both be positive, or both be negative. If (x, y) is a solution, then so is $(-x, -y)$. Therefore, we may for now assume that x and y are positive, and at the end, add for each solution (x, y) the pair $(-x, -y)$ to the list of solutions.

Combining $2xy = a^2$ and $x^2+y^2 = p$ yields $(x+y)^2 = x^2+y^2+2xy = p+a^2$. By bringing a^2 to the other side, we find

$$p = (x + y)^2 - a^2 = (x + y + a)(x + y - a).$$

Since $x + y + a$ is positive, also $x + y - a$ must be positive. The prime number p can be written as a product of two positive integers in only two ways: $1 \cdot p$ and $p \cdot 1$. Since $x + y + a \geq x + y - a$, we obtain $x + y + a = p$ and $x + y - a = 1$.

Adding these two equations, we get $2x + 2y = p + 1$. We also know that $x^2 + y^2 + 1 = p + 1$, so $2x + 2y = x^2 + y^2 + 1$. By bringing all terms to the right-hand side and adding 1 to both sides, we obtain

$$1 = x^2 + y^2 - 2x - 2y + 2 = (x - 1)^2 + (y - 1)^2.$$

We now have two perfect squares that add up to 1. This implies that one of the squares is 0 and the other is 1. So $(x - 1)^2 = 0$ and $(y - 1)^2 = 1$, or $(x - 1)^2 = 1$ and $(y - 1)^2 = 0$. As x and y are positive, we find two possible solutions: $x = 1$ and $y = 2$, or $x = 2$ and $y = 1$. In both cases $2xy = 4$ is a perfect square and $x^2 + y^2 = 5$ is a prime number. It follows that both are indeed solutions.

Adding the solutions obtained by replacing (x, y) by $(-x, -y)$, we obtain a total of four solutions (x, y), namely

$$(1, 2), (2, 1), (-1, -2), (-2, -1).$$

5. Suppose that on a given day, Sabine is left with n^2 shells, where $n > 1$. Then the next day, she will give n shells to her sister and will be left with $n^2 - n$ shells. This is more than $(n - 1)^2$, since

$$(n - 1)^2 = n^2 - 2n + 1 = (n^2 - n) - (n - 1) < n^2 - n$$

as $n > 1$. The next day, she therefore gives $n - 1$ shells to her sister and is left with $n^2 - n - (n - 1) = (n - 1)^2$ shells, again a perfect square. We see that the numbers of shells that Sabine is left with are alternately a perfect square and a number that is not a perfect square.

Let d be the first day that Sabine is left with a number of shells that is a perfect square, say n^2 shells. Then days $d + 2, d + 4, \ldots, d + 18$ are the second to tenth day that the remaining number of shells is a perfect square (namely $(n-1)^2, (n-2)^2, \ldots, (n-9)^2$ shells). We conclude that $d+18 = 28$, and hence $d = 10$.

On day 26 the number of remaining shells is at least 1000, but on days 27 and 28 this number is less than 1000. We see that $(n-9)^2 < 1000 \leq (n-8)^2$. As $31^2 < 1000 \leq 32^2$, we see that $n - 8 = 32$, and hence $n = 40$. We find that day 10 is the first day that the number of remaining shells is a perfect square, and that this number is 40^2.

In the remainder of the proof, we will use the following observation.

Observation. On any day, starting with more shells, means that Sabine will have more (or just as many) shells left after giving shells to her sister.

Indeed, suppose that Sabine starts the day with x shells, say $n^2 \leq x < (n+1)^2$. After giving away shells, she will be left with $x - n$ shells. If she had started with $x + 1$ shells instead of x, she would have been left with $x + 1 - n > x - n$ or $x + 1 - (n + 1) = x - n$ shells.

Let x be the number of shells remaining on day 8. The obvious guess $x = 41^2 = 1681$ is incorrect as x cannot be a perfect square. We therefore try $x = 41^2 - 2$, $x = 41^2 - 1$, and $x = 41^2 + 1$. The table shows the number of shells remaining on day 8, 9, and 10.

day 8	day 9	day 10
$41^2 - 2 = 1679$	$1679 - 40 = 1639$	$1639 - 40 = 1599$
$41^2 - 1 = 1680$	$1680 - 40 = 1640$	$1640 - 40 = 1600$
$41^2 + 1 = 1682$	$1682 - 41 = 1641$	$1641 - 40 = 1601$

We see that the case $x = 1679$ is ruled out because it would imply that fewer than $40^2 = 1600$ shells are left on day 10. By the above observation, this also rules out the case $x < 1679$. The case $x = 1682$ is ruled out because it would imply that more than 40^2 shells will be left on day 10. Hence, also $x > 1682$ is ruled out. The number of shells left on day 8 must therefore be $41^2 - 1$.

To follow the pattern back in time, we consider the case that the number of remaining shells is just shy of a perfect square. Suppose that on a given day the number of remaining shells is $n^2 - a$, where $1 \leq a < n$. Then the following day, the number of remaining shells is $n^2 - a - (n - 1)$. Since $a < n$, we have $n^2 - a - (n-1) > n^2 - n - (n-1) = (n-1)^2$. The day after that, the number of remaining shells must therefore be $n^2 - a - (n-1) - (n-1) = (n-1)^2 - (a-1)$.

So if Sabine originally had $45^2 - 5$ shells, then the number of remaining shells on days 2, 4, 6, and 8 are $44^2 - 4$, $43^2 - 3$, $42^2 - 2$, and $41^2 - 1$, respectively. This gives us a solution.

If Sabine originally had $45^2 - 4$ shells, then she would be left with too many shells on day 8, namely $41^2 - 0$. The original number of shells could therefore not have been $45^2 - 4$ or more.

If Sabine originally had $45^2 - 6$ shells, then she would be left with too few shells on day 8, namely $41^2 - 2$. The original number of shells could therefore not have been $45^2 - 6$ or fewer.

We conclude that the only possibility is that Sabine started with a collection of $45^2 - 5 = 2020$ shells.

BxMO Team Selection Test, March 2021

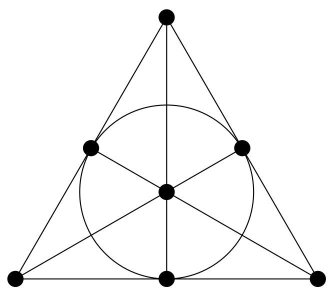
Problems

1. Let $ABCD$ be a cyclic quadrilateral with $|AB| = |BC|$. Point E lies on the arc CD which does not contain A and B. The intersection of BE and CD is denoted by P, the intersection of AE and BD is denoted by Q. Prove that $PQ \parallel AC$.

2. Determine all triples (x, y, z) of real numbers satisfying:

$$\begin{aligned} x^2 - yz &= |y - z| + 1, \\ y^2 - zx &= |z - x| + 1, \\ z^2 - xy &= |x - y| + 1. \end{aligned}$$

3. Let p be a prime number greater than 2. Patricia wants to assign 7 not necessarily distinct numbers of $\{1, 2, \ldots, p\}$ to the black dots in the figure below, in such a way that the product of three numbers on a line or circle always gives the same remainder upon divison by p.



(a) Suppose Patricia is using the number p at least once. How often does she need to use the number p at least?

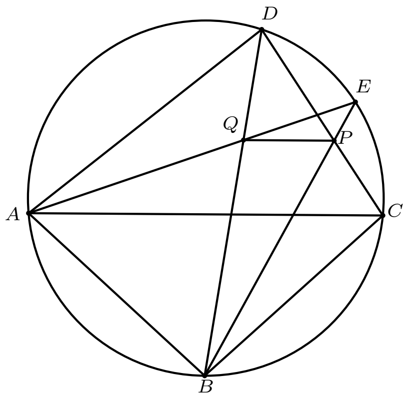
(b) Suppose that Patricia does not use the number p. In how many ways can she assign the numbers? (Two ways are considered to be different if to at least one black dot distinct numbers have been assigned. The figure is not being turned or mirrored.)

4. Jesse and Tjeerd are playing a game. Jesse has $n \geq 2$ stones. There are two boxes: in the black box there is space for half of the stones (rounded down) and in the white box there is space for half of the stones (rounded up). Jesse and Tjeerd alternate turns, with Jesse as first player. In his turn, Jesse takes one new stone, writes a positive real number on the stone and puts it in one of the boxes which is not full yet. Tjeerd can see all the numbers on the stones in each of the boxes and is allowed to move one stone of his choice to the other box, if that other box is not full yet, but he is also allowed to choose to do nothing. The game stops when both boxes are full. If the total value of the stones in the black box is greater than the total value of the stones in the white box, Jesse wins; otherwise Tjeerd wins. Determine for each $n \geq 2$ who can always win this game (and give a winning strategy).

5. A triangle ABC has the property that $|AB| + |AC| = 3|BC|$. Let T be the point on line segment AC satisfying $|AC| = 4|AT|$. Let K and L be points on the interior of line segments AB and AC, respectively, such that $KL \parallel BC$, and KL is tangent to the incircle of $\triangle ABC$. Let S be the intersection of BT and KL. Determine the ratio $\frac{|SL|}{|KL|}$.

Solutions

1. Because $|AB| = |BC|$, we have $\angle AEB = \angle BDC$, hence $\angle QEP = \angle AEB = \angle BDC = \angle QDP$, which yields that $QPED$ is a cyclic quadrilateral. Therefore, $\angle QPD = \angle QED = \angle AED = \angle ACD$. From this, we get that QP and AC are parallel. $\square$



2. The system of equations is symmetric: if you swap x and y, for example, then the third equation stays the same and the first two equations are swapped. Hence, we can assume without loss of generality that $x \geq y \geq z$. Then the system of equations becomes:

$$\begin{aligned} x^2 - yz &= y - z + 1, \\ y^2 - zx &= x - z + 1, \\ z^2 - xy &= x - y + 1. \end{aligned}$$

Subtracting the second equation from the first, we obtain $x^2 - y^2 + z(x-y) = y - x$, or $(x-y)(x+y+z+1) = 0$. This yields $x = y$ or $x+y+z = -1$. Subtracting the third equation from the second, we obtain $y^2 - z^2 + x(y-z) = y - z$, or $(y-z)(y+z+x-1) = 0$. This yields $y = z$ or $x + y + z = 1$.

We now distinguish two cases: $x = y$ and $x \neq y$. In the first case, we have $y \neq z$, as otherwise we would have $x = y = z$ for which the first equation becomes $0 = 1$, a contradiction. Now it follows that $x + y + z = 1$, or $2x + z = 1$. Substituting $y = x$ and $z = 1 - 2x$ in the first equation yields $x^2 - x(1-2x) = x - (1-2x) + 1$, which can be simplified to $3x^2 - x = 3x$, or $3x^2 = 4x$. We get $x = 0$ or $x = \frac{4}{3}$. With $x = 0$, we find $y = 0$, $z = 1$, but does not satisfy our assumption $x \geq y \geq z$. Thus, the only remaining possibility is $x = \frac{4}{3}$, which gives the triple $(\frac{4}{3}, \frac{4}{3}, -\frac{5}{3})$. We verify that this is indeed a solution.

Now consider the case $x \neq y$. Then we have $x + y + z = -1$, hence we cannot have $x + y + z = 1$, and we see that $y = z$. Now $x + y + z = -1$ yields $x + 2z = -1$, hence $x = -1 - 2z$. Now the first equality becomes $(-1 - 2z)^2 - z^2 = 1$, which can be simplified to $3z^2 + 4z = 0$. From this, we conclude that $z = 0$ or $z = -\frac{4}{3}$. With $z = 0$, we find $y = 0$, $x = -1$, which does not satisfy our assumption $x \geq y \geq z$. Hence, the only remaining possibility is $z = -\frac{4}{3}$, and this gives rise to the triple $(\frac{5}{3}, -\frac{4}{3}, -\frac{4}{3})$. We verify that this is indeed a solution.

By also considering the permutations of these two solutions, we find all six solutions: $(\frac{4}{3}, \frac{4}{3}, -\frac{5}{3})$, $(\frac{4}{3}, -\frac{5}{3}, \frac{4}{3})$, $(-\frac{5}{3}, \frac{4}{3}, \frac{4}{3})$, $(\frac{5}{3}, -\frac{4}{3}, -\frac{4}{3})$, $(-\frac{4}{3}, \frac{5}{3}, -\frac{4}{3})$, and $(-\frac{4}{3}, -\frac{4}{3}, \frac{5}{3})$. □

3. (a) From now on, with *'lines'* we mean the six lines and the circle. As soon as the number p is used somewhere, there is a 'line' whose product is divisible by p, and hence has remainder 0 upon division by p. All 'lines' must give remainder 0 in that case, hence all 'lines' contain at least one p. This can be achieved by assigning the number p to the bottom three dots. Suppose that it is already possible with at most two times the number p. Each dot is lying on exactly three 'lines', hence there are at most six 'lines' containing a p. There are seven 'lines' in total, however, hence this is impossible. We conclude that she needs the number p at least three times in total.

(b) Denote the bottom three numbers from left to right by a, b, and c. Denote the number in the middle on the left side of the triangle by d. Now we can compute all numbers modulo p; note that all numbers are invertible modulo p because p itself does not appear, and p is prime. The bottom line has product abc. The left side also needs to have this product, hence the top number in the triangle must be congruent to $abc(ad)^{-1} = bcd^{-1}$ modulo p. If we consider the line from the right bottom to the left middle, we see that the middle number must be congruent to $abc(cd)^{-1} = abd^{-1}$. By considering the circle, we find the number in the middle on the right side: $abc(bd)^{-1} = acd^{-1}$.

On the right side, we now get the equation $bcd^{-1} \cdot acd^{-1} \cdot c \equiv abc \mod p$, hence $c^2 \equiv d^2 \mod p$. Similarly, the vertical line yields $b^2 \equiv d^2$, and the line from the left bottom to the right middle yields $a^2 \equiv d^2 \mod p$. We conclude that the numbers a, b, c, and d all must have the same square. From $x^2 \equiv y^2$ we obtain $(x - y)(x + y) \equiv 0 \mod p$, hence $p \mid x - y$ or $p \mid x + y$, because p is prime. Therefore, we have

$x \equiv y$ or $x \equiv -y$. These are two possibilities, because if $y \equiv -y$ were to hold, then $2y \equiv 0$, hence $p \mid 2y$, hence $p \mid y$ since $p > 2$; a contradiction. Thus the numbers b, c, and d are all congruent to a or $-a$. If these conditions are met, then we do have $a^2 \equiv b^2 \equiv c^2 \equiv d^2$ and from the preceding argument, we get that the product of the three numbers on all six lines and the circle is congruent to abc.

There are $p-1$ possibilities for the number a, and after choosing a, there are 2 possibilities for each of the numbers b, c, and d. In total, there are $8(p-1)$ ways to assign the numbers. □

4. We will show that the capicity of the two boxes does not matter, as long as the total capacity is n (and at least 1 for each box). Jesse can always win this game, and can do that by first playing the power $2^0 = 1$ of two, and then in each following turn the next power of two that is smaller or greater. That means: if he played the numbers

$$2^{-i}, 2^{-(i-1)}, \ldots, 2^{-1}, 2^0, 2^1, \ldots, 2^{j-1}, 2^j$$

at a certain moment, he will play either $2^{-(i+1)}$ or 2^{j+1} in his next turn.

By playing cleverly, Jesse can make sure that the greatest power of two among the stones played so far is always contained in the black box. We will prove this by induction. In his first move, he puts the stone with value 2^0 in the black box and the claim is true; this is the base case of the induction. When it is his turn again, and Tjeerd moved the greatest power of two so far, which according to the induction hypothesis was contained in the black box, to the white box, then the black box actually has a free space, and Jesse can put a new greater power of two in there, and the claim is true. If Tjeerd moved some other stone or did nothing, then the greatest power of two so far is still in the black box, and Jesse can play a smaller power of two; it does not matter where he puts it. Also in this case, the claim is true. This proves the induction step, and the claim is proved.

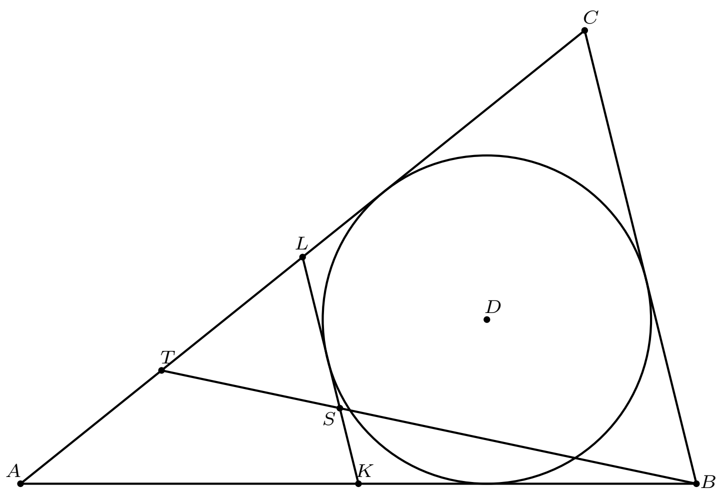
Therefore, after playing the last stone, the greatest power of two is in the black box. It is greater than the sum of all smaller powers of two played ($2^j > 2^j - 2^{-i} = 2^{j-1} + 2^{j-2} + \cdots + 2^{-(i-1)} + 2^{-i}$), hence it is certainly greater than the sum of the powers of two in the white box. Therefore, the total value inside the black box is greater than the total value in the white box. □

5. Denote the radius of the incricle of $\triangle ABC$ by r. Then the area of triangle ABC is

$$\frac{1}{2}|AB| \cdot r + \frac{1}{2}|BC| \cdot r + \frac{1}{2}|AC| = \frac{1}{2}r \cdot (3|BC| + |BC|) = 2r|BC|.$$

On the other hand, the area of ABC equals $\frac{1}{2}h|BC|$, where h is the altidude from A. Hence, $h = 4r$. Because the distance from KL to BC is exactly $2r$, the distance from A to KL is also $2r$. Triangles AKL and ABC are similar, because $KL \parallel BC$, and the altitudes from A have lengths $2r$ and $4r$, respectively, giving a multiplication factor of exactly 2. Hence, K is the midpoint AB, and L is the midpoint of AC.

For the point T, we have $|AC| = 4|AT|$, hence $|AT| = \frac{1}{4}|AC| = \frac{1}{2}|AL|$, hence T is the midpoint of AL. Now consider triangle ABL. In this triangle, the segment BT is a median, because T is the midpoint of AL. Also LK is a median as K is the midpoint AB. Their intersection point S is the centroid, from which we get that $\frac{|SL|}{|KL|} = \frac{2}{3}$. □



IMO Team Selection Test 1, June 2021

Problems

1. The sequence $a_0, a_1, a_2, \ldots$ of integers is defined by $a_0 = 3$ and

$$a_{n+1} - a_n = n(a_n - 1)$$

for all $n \geq 0$. Determine all integers $m \geq 2$ for which $\gcd(m, a_n) = 1$ for all $n \geq 0$.

2. Find all quadruples (x_1, x_2, x_3, x_4) of real numbers which are solutions of the following system of six equations:

$$\begin{aligned} x_1 + x_2 &= x_3^2 + x_4^2 + 6x_3x_4, \\ x_1 + x_3 &= x_2^2 + x_4^2 + 6x_2x_4, \\ x_1 + x_4 &= x_2^2 + x_3^2 + 6x_2x_3, \\ x_2 + x_3 &= x_1^2 + x_4^2 + 6x_1x_4, \\ x_2 + x_4 &= x_1^2 + x_3^2 + 6x_1x_3, \\ x_3 + x_4 &= x_1^2 + x_2^2 + 6x_1x_2. \end{aligned}$$

3. Let ABC be an acute non-isosceles triangle with orthocentre H. Let O be the circumcentre of triangle ABC, and let K be the circumcentre of triangle AHO. Prove that the reflection of K in OH lies on BC.

4. On a rectangular board consisting of $m \times n$ squares ($m, n \geq 3$), dominos have been placed (2×1- or 1×2-tiles), not overlapping each other. Each domino covers exactly two squares of the board. Suppose that the placement of the dominos has the property that no extra domino can be placed on the board, and the four corners of the board are not all empty. Prove that at least $\frac{2}{3}$ of the squares of the board is covered by dominos.

Solutions

1. The sequence is given by the formula $a_n = 2 \cdot n! + 1$ for $n \geq 0$. (We use the usual definition $0! = 1$, which satisfies $1! = 1 \cdot 0!$, in the same way we have $n! = n \cdot (n-1)!$ for other positive integers n.) We will prove the equality by induction. We have $a_0 = 3$, which equals $2 \cdot 0! + 1$. Now suppose for certain $k \geq 0$ that $a_k = 2 \cdot k! + 1$, then

$$a_{k+1} = a_k + k(a_k - 1) = 2 \cdot k! + 1 + k \cdot 2 \cdot k! = 2 \cdot k! \cdot (1+k) + 1 = 2 \cdot (k+1)! + 1.$$

This finishes the induction.

We see that a_n is always odd, hence $\gcd(2, a_n) = 1$ for all n. It follows also that $\gcd(2^i, a_n) = 1$ for all $i \geq 1$. Hence, $m = 2^i$ with $i \geq 1$ satisfies the condition. Now consider an $m \geq 2$ which is not a power of two. Then m has an odd prime divisor, say p. We will show that p is a divisor of a_{p-3}. By Wilson's theorem, we have $(p-1)! \equiv -1 \mod p$. Hence,

$$\begin{aligned} 2 \cdot (p-3)! &\equiv 2 \cdot (p-1)! \cdot \big((p-2)(p-1)\big)^{-1} \\ &\equiv 2 \cdot -1 \cdot (-2 \cdot -1)^{-1} \equiv 2 \cdot -1 \cdot 2^{-1} \equiv -1 \mod p. \end{aligned}$$

So indeed we have $a_{p-3} = 2 \cdot (p-3)! + 1 \equiv 0 \mod p$. We conclude that m does not satisfy the condition. Hence, the only values of m satisfying the condition are powers of two. □

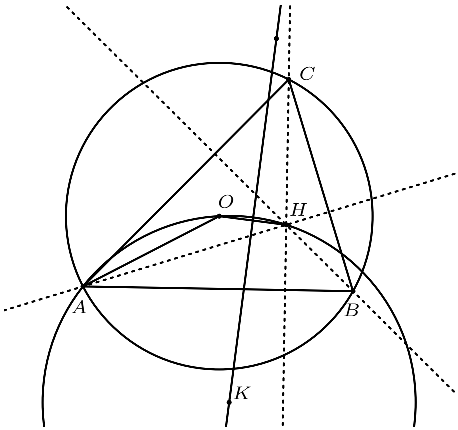
2. Subtracting the second equation from the first yields $x_2 - x_3 = x_3^2 - x_2^2 + 6x_4(x_3 - x_2)$, which we can factor as $0 = (x_3 - x_2)(x_3 + x_2 + 1 + 6x_4)$. We see that $x_2 = x_3$ or $x_2 + x_3 + 1 + 6x_4 = 0$. Similarly, we also have either $x_2 = x_3$ or $x_2 + x_3 + 1 + 6x_1 = 0$. Hence, if $x_2 \neq x_3$, the second equality must hold in both cases; subtracting one from the other, we obtain $x_1 = x_4$. We conlude that either $x_2 = x_3$ or $x_1 = x_4$. Analogously, we get for each permutation (i, j, k, l) of $(1, 2, 3, 4)$ that either $x_i = x_j$ or $x_k = x_l$.

We will prove that at least three of the x_i must be equal. If all four are equal, this is true of course. Otherwise, there are two unequal ones, say $x_1 \neq x_2$ without loss of generality. Then we have $x_3 = x_4$. If also $x_1 = x_3$ holds, then there are three equal elements. Otherwise, we have $x_1 \neq x_3$, hence $x_2 = x_4$ and we also get three equal elements. Up to order, the quadruple (x_1, x_2, x_3, x_4) is thus equal to a quadruple of the shape (x, x, x, y), where we could have that $x = y$.

Substituting this in the equations gives $x + y = 8x^2$ and $2x = x^2 + y^2 + 6xy$. Adding these two equations: $3x + y = 9x^2 + y^2 + 6xy$. The right hand side can be factored as $(3x + y)^2$. Defining $s = 3x + y$, the equation becomes $s = s^2$, from which we get either $s = 0$ or $s = 1$. We have $s = 3x + y = 2x + (x + y) = 2x + 8x^2$. Hence, $8x^2 + 2x = 0$ or $8x^2 + 2x = 1$.

In the first case, we have $x = 0$ or $x = -\frac{1}{4}$. We find $y = 0 - 3x = 0$ and $y = 0 - 3x = \frac{3}{4}$, respectively. In the second case, we get the factorisation $(4x - 1)(2x + 1) = 0$, hence $x = \frac{1}{4}$ or $x = -\frac{1}{2}$. We find $y = 1 - 3x = \frac{1}{4}$ or $y = 1 - 3x = \frac{5}{2}$, respectively.

Altogether, we found the following quadruples: $(0, 0, 0, 0)$, $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{5}{2})$, and permutations thereof. It is a simple computation to verify that all these quadruples are indeed solutions to the equations. □



3. We consider the configuration as in the figure. Other configurations are treated analogously. Denote by D the second intersection of AH with the circumcircle of $\triangle ABC$. Denote by S the second intersection of the circumcircles of ABC and AHO. (Because $\triangle ABC$ is acute, both O and H lie in the interior of ABC and also in the interior of the circumcircle, hence D and S both exist.)

We have

$$\angle OSH = \angle OAH = \angle OAD = \angle ODA = \angle ODH,$$

where we use that $|OA| = |OD|$. Moreover, we have

$$\angle OHD = 180^\circ - \angle OHA = 180^\circ - \angle OSA = 180^\circ - \angle OAS = \angle OHS,$$

where we use that $|OA| = |OS|$. Now we conclude that $\triangle OHS \cong \triangle OHD$ (SAA). This yields that D and S are each others reflection images in OH.

Therefore, if we reflect the circumcentre K of $\triangle OHS$ in OH, we get the circumcentre L of $\triangle OHD$. Now we must prove that L lies on BC.

Point D is the reflection of H in BC. This is a known fact, which we can prove as follows: $\angle DBC = \angle DAC = \angle HAC = 90° - \angle ACB = \angle HBC$ and analogously $\angle DCB = \angle HCB$, hence $\triangle DBC \cong \triangle HBC$ (ASA). Hence, D is indeed the reflection of H in BC, from which we get that BC is the perpendicular bisector of HD. Becuase L lies on the perpendicular bisector of HD, we get that L lies on BC, which is what we wanted to prove. □

4. Assign each empty square to the domino directly right of this square (unless the square is on the right edge of the board). Now suppose that two empty squares are assigned to the same domino, then this domino must be placed vertically and both squares left of this domino are empty. However, that would mean that another domino could fit, which is a contradiction. Hence, no two empty squares are assigned to the same domino.

The empty squares on the right edge of the board have not been assigned a domino yet. We try to assign these squares to dominos that do not have an empty square directly left of them (i.e. dominos which have not been assigned yet). First suppose that we succeed in assigning all empty squares on the right edge of the board in this way to different dominos. In that case, we assigned each empty square to a domino, where no domino has been assigned more than one square. Because each domino covers two squares of the board, there are two covered squares for each empty square, and hence at most $\frac{1}{3}$ of the squares is uncovered. In this case, we are done.

Now we will show that this assignment always works. Let k be the number of empty squares on the right edge, and ℓ the number of empty squares on the left edge. The empty squares on the left edge cannot be adjacent, hence there are at least $\ell - 1$ dominos on the left edge and these all do not have an empty square left of them. If $\ell > k$, then there are enough dominos on the left edge to assign to all empty squares on the right edge. If $\ell < k$, we could turn everything around and assign all empty squares to the domino left of them and we could also prove that at most $\frac{1}{3}$ of the squares on the board are uncovered. The only remaining situation is when $\ell = k$ and both on the left and right exactly $k-1$ dominos are on the edge. For both edges, we have that there must be an empty square between each two dominos, and there must also be empty squares in the corners. This, however, is in contradiction with the condition that not all corners are empty. Hence, this situation cannot occur. □

IMO Team Selection Test 2, June 2021

Problems

1. Let Γ be the circumcircle of a triangle ABC and let D be a point on segment BC. The circle that passes through B and D and is tangent to Γ and the circle that passes through C and D and is tangent to Γ, intersect at a point $E \neq D$. The line DE intersects Γ at two points, X and Y. Prove that $|EX| = |EY|$.

2. Prickle and Sting are playing a game on an $m \times n$-board, where m and n are positive integers. They alternatingly take turns, and Prickle goes first. Prickle must, during his turn, place a pawn on a square which doesn't contain a pawn yet. Sting must, during his turn, also place a pawn on a square which doesn't contain a pawn yet, but additionally, his pawn must be placed in a square that is adjacent to the square in which Prickle placed his pawn the previous turn.

Sting wins once the entire board is completely filled with pawns. Prickle wins if Sting cannot place a pawn in his turn, while there is at least one empty square on the board.

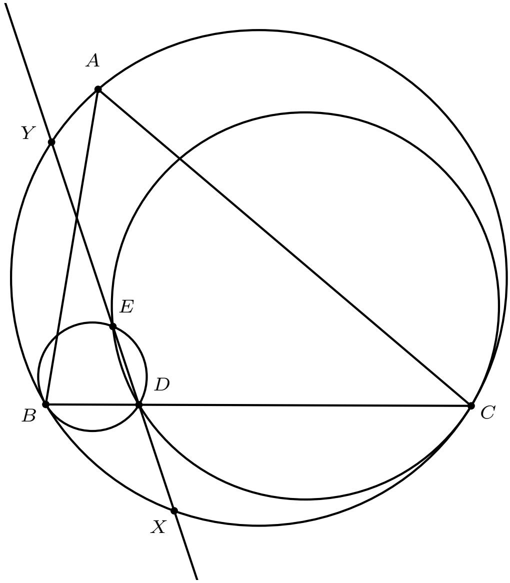
Determine for all pairs (m, n) of positive integers which of Prickle and Sting has a winning strategy.

3. Show that for every positive integer n there exist positive integers a and b with

$$n \mid 4a^2 + 9b^2 - 1.$$

4. Determine all positive integers n with the following property: for every triple (a, b, c) of positive real numbers there exists a triple (k, ℓ, m) of non-negative integers such that an^k, bn^ℓ, cn^m are the lengths of sides of a (non-degenerate) triangle.

Solutions



1. We consider the configuration as in the figure, where E is at least as close to B as it is to C. The proof in the case of the configuration in which this is the other way around, is analogous.

Let O be the centre of Γ. The angle between the line BC and the common tangent in B is on the one hand, by the inscribed angle theorem (tangent case), equal to $\angle BED$, and on the other hand equal to $\angle BAC$. So $\angle BED = \angle BAC$. Analogously, we show that $\angle CED = \angle BAC$, so $\angle BEC = \angle BED + \angle CED = 2\angle BAC = \angle BOC$, where we use the inscribed angle theorem to derive the last step. Therefore E lies on the circle that passes through B, O, and C. If $E = O$, then we're done, as $|EX|$ and $|EY|$ then both are the radius of the circle.

So suppose that $E \neq O$, then in the configuration considered, $BEOC$ is a cyclic quadrilateral. Then $\angle BEO = 180° - \angle BCO$. In the isosceles triangle BOC, we have $\angle BCO = 90° - \frac{1}{2}\angle BOC = 90°\angle BAC$, so $\angle BEO = 180° - (90° - \angle BAC) = 90° + \angle BAC$. Hence $\angle DEO = \angle BEO - \angle BED = 90° + \angle BAC - \angle BAC = 90°$. Therefore EO is perpendicular to DE and therefore also perpendicular to chord XY, from which follows that E is the midpoint of XY. We conclude that $|EX| = |EY|$. □

2. We use the convention that m is the number of rows, and that n is the number of columns. If m is even, then we pair the squares of the board as follows: in every column we pair the top two squares, then squares 3 and 4, etc. As the number of rows is even, this pairs the squares of every column completely. Sting can use the following strategy: whenever Prickle places a pawn into one of a pair of squares, Sting places a pawn into the other, which is necessary adjacent to the pawn that Prickle just placed. After each move of Sting, all pairs contain either zero or two pawns, so Sting can always place a pawn according to this strategy. Therefore Sting can make sure that the entire board is eventually completely filled with pawns, so Sting wins. Analogously, if n is even, then Sting has a similar winning strategy.

If $m = n = 1$, Sting wins after Prickle's first move. If $m = 1$ and $n = 3$ (or the other way around), Sting can place his first pawn on a square adjacent to the square in which Prickle has just placed his; such an adjacent (empty) square always exists. The board will then be completely filled after Prickle's next move, so Sting has a winning strategy in these cases as well.

Now consider the case $m = n = 3$. Prickle can follow the following strategy. He places his first pawn in the centre square. Sting must place his pawn either in the same row or in the same column. Without loss of generality, we assume that Sting's pawn is placed in the same column as that of Prickle. Prickle then places a pawn in the remaining square in the middle column. At that point, the left and right columns are completely empty. Sting must place his pawn into one of these columns. Prickle then chooses any square in the other column, all three squares of which are still empty. This forces Sting to place a pawn in that column as well, as the middle column was already completely filled. Now Prickle places a pawn in the only remaining square of that column, after which Sting can no longer place a pawn, while there are two empty squares remaining. Therefore Prickle has a winning strategy if $m = n = 3$.

The remaining case is that m and n are both odd, and that at least one of m and n is at least 5. We consider the case that n (the number of columns) is at least 5; the other case is analogous. Prickle can follow the following strategy. Prickle places his pawns into the centre column until it is completely filled with pawns. When it is Prickle's turn again, there is an even number of pawns on the board, all in the three columns in the centre of the board, of which at least the middle one is completely filled. The first and the last column are completely empty. As there is an odd number of squares remaining, either the area to the left of the middle column has an odd number of empty squares remaining, or the area to the right has. Prickle chooses the area which contains an odd number of empty squares, and places his pawns there until that area is full. As the centre column is completely filled, Sting must also place all of his pawns in that area. As this area contained an odd number of empty squares, Prickle is the last player (to be able) to place a pawn in this area. Sting can no longer place a pawn, while the other area contains a column of empty squares, so Prickle wins. (Of course, if Sting runs out of moves earlier, Prickle also wins.)

We conclude that Prickle wins if m and n are odd and $m \geq 5$, if m and n are odd and $n \geq 5$, and if $m = n = 3$. Sting wins in all other cases: if m and n are both even, if $m = n = 1$, if $m = 1$ and $n = 3$, and if $m = 3$ and $n = 1$. □

3. If $n = 1$, all choices of a and b are solutions. Now suppose that $n > 1$ and let p be a prime divisor of n. Let k be the number of factors of p in n. We give a condition for a and b modulo p^k which guarantees that $p^k | 4a^2 + 9b^2 - 1$. By doing this for every prime divisor of n, we get a system of conditions for a and b modulo the various prime powers. Then, by the Chinese remainder theorem, there exist a and b satisfying all conditions simultaneously.

If $p \neq 2$, then we consider the condition that $2a \equiv 1 \mod p^k$ and $b \equiv 0 \mod p^k$. As 2 has a multiplicative inverse module p^k, this condition can be satisfied. We then have

$$4a^2 + 9b^2 - 1 = (2a)^2 + 9b^2 - 1 \equiv 1^2 + 9 \cdot 0 - 1 = 0 \mod p^k.$$

Therefore all a and b satisfying this condition are solutions.

If $p = 2$, then we consider the condition that $a \equiv 0 \mod 2^k$ and $3b \equiv 1 \mod 2^k$. As 3 has a multiplicative inverse modulo 2^k, this condition can be satisfied. We then have

$$4a^2 + 9b^2 - 1 = 4a^2 + (3b)^2 - 1 \equiv 4 \cdot 0 + 1^2 - 1 = 0 \mod 2^k.$$

Therefore all a and b satisfying this condition are solutions. □

4. It is clear that $n = 1$ does not satisfy the property, as not every three positive real numbers a, b, and c are the lengths of the sides of a triangle.

We first show that any $n \geq 5$ cannot satisfy the property by considering the triple $(a, b, c) = (1, 2, 3)$. Suppose that there exist k, ℓ, m such that $n^k, 2n^\ell, 3n^m$ are the lengths of the sides of a triangle. As $n \neq 2, 3$, no three of these are equal. By repeatedly removing common factors n if they exist, we can and do assume that one of k, ℓ, m is equal to 0. Suppose that the other two of those three integers are positive, then their corresponding side lengths both are multiples of n. Their difference then is at least n, while the third side has length of at most 3. This contradicts the triangle inequality. Hence of k, ℓ, m, at least two must be zero. The corresponding two side lengths then sum to at most 5, so the third side must have length less than 5. As $n \geq 5$, this third side cannot have a factor n either, so k, ℓ, m are all equal to 0. However, then 1, 2, 3 should be the lengths of the sides of a triangle, while $3 = 2 + 1$. This is a contradiction. We deduce that $n \geq 5$ does not satisfy the property.

Now consider $n = 2, 3, 4$. We construct (k, ℓ, m) as follows. Take a triple (a, b, c). If its entries already are the lengths of the sides of a triangle, we take $k = \ell = m = 0$. Otherwise there exists a triangle inequality that is not satisfied. Without loss of generality, we assume that $a \geq b + c$. Multiply the lesser of b and c with n. If the right hand side still isn't larger than a, then take the new summands, and again multiply the lesser of the two by n. So: if $a \geq n^i b + n^j c$, then we multiply the lesser of $n^i b$ and $n^j c$ by n, and repeat if the inequality still holds. This process always stops at some point, as there exists i such that $n^i > a$.

Consider the i and j such $a \geq n^i b + n^j c$, such that applying the process makes the right hand side larger than a. We assume without loss of generality that $n^i b \leq n^i c$, so we have $a < n^{i+1} b + n^j c$. We claim that we can take (k, ℓ, m) to be equal to either $(0, i+1, j)$ or $(0, i+1, j+1)$.

By definition, we have $a < n^{i+1} b + n^j c$, and we have $n^j c < n^i b + n^j c \leq a < a + n^{i+1} b$. Therefore, if $(0, i+1, j)$ doesn't satisfy the property, then we must have $n^{i+1} b \geq a + n^j c$. Moreover $n^i b \leq n^j c$, so $n^{i+1} b \leq n^{j+1} c < a + n^{j+1} c$. We also have $a < n^{i+1} b + n^{j+1} c$, so if $(0, i+1, j+1)$ does not satisfy the property, then we must have $n^{j+1} c \geq a + n^{i+1} b$. We will derive a contradiction in case both triples do not satisfy the property, i.e. in case both $n^{i+1} b \geq a + n^j c$ and $n^{j+1} c \geq a + n^{i+1} b$.

Adding these inequalities, and subtracting $n^{i+1} b$ on both sides, yields $n^{j+1} c \geq 2a + n^j c$, or equivalently $(n-1) n^j c \geq 2a$. Therefore

$$n^{i+1} b \geq a + n^j c \geq a + \frac{2a}{n-1} = \left(1 + \tfrac{2}{n-1}\right) a,$$

from which follows that

$$a \geq n^i b + n^j c \geq \tfrac{1}{n}\left(1 + \tfrac{2}{n-1}\right) a + \tfrac{2}{n-1} a = \frac{(n-1) + 2 + 2n}{n(n-1)} a = \frac{3n+1}{n(n-1)} a.$$

So $3n + 1 \leq n(n-1)$. For $n = 2, 3, 4$, this inequality reads $7 \leq 2$, $10 \leq 6$, and $13 \leq 12$, all of which are false. This is a contradiction.

Hence $n = 2, 3, 4$ are precisely the values which satisfy the property. $\square$

IMO Team Selection Test 3, June 2021

Problems

1. Let m and n be positive integers with mn even. Jetze is going to cover an $m \times n$-board (with m rows and n columns) with domino tiles, in such a way that every domino tile covers exactly two squares, domino tiles do not protrude out of the board or overlap one another, and every square is covered by a domino tile. Merlijn then is going to colour all domino tiles on the board either red or blue. Determine the smallest non-negative integer V (depending on m and n) such that Merlijn can always make sure that in each row, the number of squares covered by a red domino tile and the number of squares covered by a blue domino tile differ by at most V, no matter in what way Jetze covers the board.

2. Let ABC be a right angled triangle with $\angle C = 90°$ and let D be the foot of the altitude from C. Let E be the centroid of triangle ACD and let F be the centroid of triangle BCD. Let P be the point satisfying $\angle CEP = 90°$ and $|CP| = |AP|$, and let Q be the point satisfying $\angle CFQ = 90°$ and $|CQ| = |BQ|$. Show that PQ passes through the centroid of triangle ABC.

3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f\big(x + yf(x+y)\big) = y^2 + f(x)f(y)$$

for all $x, y \in \mathbb{R}$.

4. Let $p > 10$ be a prime number. Show that there exist positive integers m and n with $m + n < p$ for which p is a divisor of $5^m 7^n - 1$.

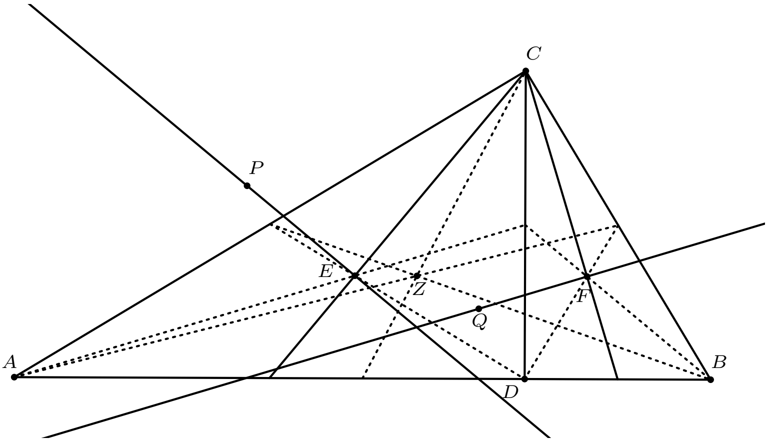
Solutions

1. First suppose that n is odd. Then we must have $V \geq 1$, as the difference must be odd. We show that $V = 1$ is always possible. Colour the vertical domino tiles in the odd numbered columns red and the vertical domino tiles in the even numbered columns blue. As in every row, every horizontal domino tile covers a square in an even numbered column and one in an odd numbered column, every row contains one more square covered by a red domino tile than squares covered by a blue domino tile. Now colour the horizontal domino tile in each row alternatingly blue and red (starting with blue). If the number of horizontal domino tiles is even, then at the end, the number of red squares will be one more than that of blue squares; if the number of horizontal domino tiles is odd, then at the end, the number of blue squares will be one more than that of red squares. The difference will therefore always be equal to 1.

Now suppose that $n \equiv 2 \mod 4$. Then we have $V \geq 2$ if Jetze places every domino tile horizontally; then every row contains an odd number of horizontal domino tiles. We show that $V = 2$ is always possible. Use the same strategy as in the odd case. After colouring the vertical domino tiles, the numbers of red and blue squares are equal. If we alternatingly colour the horizontal domino tiles in each row blue and red again, we see that in the end, in every row the difference between the number of red and blue squares is 0 or 2.

Finally, suppose that $n \equiv 0 \mod 4$. We show that $V = 0$ is always possible. Number the rows from top to bottom from 1 up to m, and let b_i be the number of vertical domino tiles of which the top square is in row i. By induction on i, we easily show that b_i is even, using the fact that a horizontal domino tile always covers an even number of squares in a row. We now colour the vertical domino tiles in rows i and $i + 1$ as follows: if $b_i \equiv 0 \mod 4$, we colour half of them red, and the other half blue, and if $b_i \equiv 2 \mod 4$ we colour two more domino tiles red than we colour blue if i is even, and we colour two more domino tiles blue than we colour red if i is odd. We show that we can now colour the horizontal domino tiles in each row k in such a way that every row has the same number of red and blue squares. If $b_{k-1} \equiv b_k \equiv 0 \mod 4$, then vertical domino tiles in row k cover the same number of red squares as blue squares. Moreover, the number of horizontal domino tiles in row k is even, so we can simply colour half of them red and half of them blue. If $b_{k-1} \equiv b_k \equiv 2 \mod 4$, then vertical domino tiles in row k again cover the same number of red squares as blue squares, since of $k - 1$ and k, one is odd and one is even. Again, the number of horizontal domino tiles in row k is even, so we can again simply colour half of them red and half of them blue. If $b_{k-1} \not\equiv b_k \mod 4$, then the difference in the number of squares covered by red vertical domino tiles and blue vertical ones is equal to 2. The number of horizontal domino tiles is odd, so we can colour those in such a way that in the end, the number of red and blue squares are equal.

Hence the minimal values for V are: $V = 1$ if n is odd, $V = 2$ if $n \equiv 2 \mod 4$, and $V = 0$ if $n \equiv 0 \mod 4$. □



2. Let M, N, R, S be the midpoints of line segments BC, CA, BD, AD. Let Z be the centroid of $\triangle ABC$. Quadrilateral $QFMC$ is cyclic as $\angle QFC = 90° = \angle QMC$. Note that therefore CQ is a diameter of the circumcircle of $QFMC$. Analogously, we see that $PNEC$ is a cyclic quadrilateral, with diameter CP.

We show that Z also lies on the circumcircles of these cyclic quadrilaterals. The similarity transforming triangle BCA into triangle BDC transforms triangle CZM into DFR, as C is mapped to D, the centroid Z is mapped to the centroid F, and the midpoint M of BC is mapped to the midpoint of BD, which is R. So $\triangle CZM \sim \triangle DFR$, in particular $\angle CZM = \angle DFR = \angle CFM$ (opposite angles). Therefore Z lies on the circumcircle of the cyclic quadrilateral $QFMC$. Analogously, we have $\angle CZN = \angle DES = \angle CEN$, from which follows that Z lies on the circumcircle of the cyclic quadrilateral $PNEC$.

We can now show that Z lies on PQ. As CQ is a diameter of the circle through Q, F, M, C, Z, we have $\angle QZC = 90°$. As CP is a diameter of the circle through P, N, E, Z, C, we also have $\angle CZP = 90°$. Hence P, Z, and Q are collinear. □

3. Note that the function $f(x) = 0$ for all x does not satisfy the condition. Hence there is some a with $f(a) \neq 0$. Substituting $x = a$ and $y = 0$, we get $f(a) = f(a)f(0)$, so $f(0) = 1$. Substituting $x = 1$ and $y = -1$, we get $f\big(1 - f(0)\big) = 1 + f(1)f(-1)$. As $f(0) = 1$, this reads $1 = 1 + f(1)f(-1)$, so $f(1) = 0$ or $f(-1) = 0$. We consider these two cases separately.

First suppose that $f(1) = 0$. Substituting $x = t$ and $y = 1-t$, and $x = 1-t$ and $y = t$ then gives

$$f(t) = (1-t)^2 + f(t)f(1-t),$$
$$f(1-t) = t^2 + f(t)f(1-t).$$

Subtracting these two gives $f(t) - f(1-t) = (1-t)^2 - t^2 = 1 - 2t$, so $f(1-t) = f(t) + 2t - 1$. Substituting this expression into the first of the above two equations gives

$$f(t) = (1-t)^2 + f(t)^2 + (2t-1)f(t),$$

or equivalently

$$f(t)^2 + (2t-2)f(t) + (1-t)^2 = 0,$$

or equivalently, $\big(f(t) - (1-t)\big)^2 = 0$. Therefore $f(t) = 1 - t$ for all t.

We verify this function in the original equation: the left hand side is $1 - (x + y(1-x-y)) = 1 - (x + y - xy - y^2) = 1 - x - y + xy + y^2$, and the right hand side is $y^2 + (1-x)(1-y) = y^2 + 1 - x - y + xy$, which is equal to the left hand side. Therefore the function $f(x) = 1 - x$ for all x is a solution to the original equation.

Now suppose that $f(-1) = 0$. Substituting $x = t$ and $y = -1-t$, and $x = -1-t$ and $y = t$, then gives

$$f(t) = (-1-t)^2 + f(t)f(-1-t),$$
$$f(-1-t) = t^2 + f(t)f(-1-t).$$

Subtracting these two gives $f(t) - f(-1-t) = (-1-t)^2 - t^2 = 1 + 2t$, so $f(-1-t) = f(t) - 2t - 1$. Substituting this expression into the first of the above two equations gives

$$f(t) = (-1-t)^2 + f(t)^2 + (-2t-1)f(t),$$

or equivalently

$$f(t)^2 - (2t+2)f(t) + (t+1)^2 = 0,$$

or equivalently, $\big(f(t) - (t+1)\big)^2 = 0$. We deduce that $f(t) = t + 1$ for all t.

We verify this function satisfies the original equation. The left hand side is $x + y(x + y + 1) + 1 = x + xy + y^2 + y + 1$ and the right hand side is $y^2 + (x+1)(y+1) = y^2 + xy + x + y + 1$, which is equal to the left hand side. Hence the function $f(x) = x + 1$ for all x satisfies the original equation.

We conclude that there are exactly two solutions of the original equation: $f(x) = 1 - x$ for all x, and $f(x) = x + 1$ for all x. $\square$

4. By Fermat's Little Theorem, we have $a^{p-1} \equiv 1 \mod p$ for all a such that $p \nmid a$. As $p > 10$, p is odd, so $p - 1$ is even. We have

$$\left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} + 1\right) = a^{p-1} - 1 \equiv 0 \mod p.$$

So $p \mid (a^{\frac{p-1}{2}} - 1)(a^{\frac{p-1}{2}} + 1)$, so p is a divisor of at least one of the factors. Hence $a^{\frac{p-1}{2}}$ is congruent to 1 or -1 modulo p.

We apply this to $a = 5$ and $a = 7$. Note that $p > 10$, so $p \neq 5, 7$. If $5^{\frac{p-1}{2}} \equiv 1 \mod p$ and $7^{\frac{p-1}{2}} \equiv 1 \mod p$, we choose $m = n = \frac{p-1}{2}$, which satisfy the condition. The same holds if $5^{\frac{p-1}{2}} \equiv -1 \mod p$ and $7^{\frac{p-1}{2}} \equiv -1 \mod p$.

The remaining case is that one is congruent to 1 and the other is congruent to -1. Assume that $5^{\frac{p-1}{2}} \equiv 1 \mod p$ and $7^{\frac{p-1}{2}} \equiv -1 \mod p$. The case in which it is the other way around is analogous.

If there exists an n such that $0 < n < \frac{p-1}{2}$ and $7^n \equiv 1 \mod p$, then we choose this n and $m = \frac{p-1}{2}$, which satisfy the condition. If not, then no n such that $\frac{p-1}{2} < n < p - 1$ and $7^n \equiv 1 \mod p$ can exist either, since otherwise $7^{p-1-n} \equiv 7^{p-1}(7^n)^{-1} \equiv 1 \mod p$, while $0 < p - 1 - n < \frac{p-1}{2}$, which is a contradiction. Moreover, no i, j such that $1 \leq i < j \leq p - 1$ and $7^i \equiv 7^j \mod p$ can exist, since otherwise $7^{j-i} \equiv 1 \mod p$ with $1 \leq j - i < p - 1$. Hence 7^i assumes distinct values for $1 \leq i \leq p - 1$ modulo p, which are all non-zero. So 7^i assumes all non-zero values modulo p for $1 \leq i \leq p - 1$.

In particular, there exists an n such that $7^n \equiv 5^{-1} \mod p$. Then $n \leq p - 2$, since $7^{p-1} \equiv 1 \not\equiv 5^{-1} \mod p$. Choosing this n and $m = 1$ gives $7^n \cdot 5^m \equiv 5^{-1} \cdot 5 \equiv 1 \mod p$.

Therefore it is always possible to find m and n satisfying the conditions. $\square$

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