58th Dutch Mathematical Olympiad 2019

NEDERLANDSE
WISKUNDE
OLYMPIADE

Contents

Introduction

The selection process for IMO 2020 started with the first round in January 2019, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 8164 students from 330 secondary schools participated.

The 940 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 118 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 58th edition 2019.

The 30 most outstanding candidates of the Dutch Mathematical Olympiad 2019 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO, which was supposed to be held in the Netherlands. Being the host country, a second team of another four girls was allowed to participate. Unfortunately, due to the Covid-19 pandemic the EGMO could not take place in Egmond aan Zee. Instead, a virtual event was held, where the Dutch team managed to win three bronze medals. For more information about the EGMO (including the 2020 paper), see www.egmo.org.

In March a selection test of three and a half hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO). This contest was turned into a virtual event as well, held on 2 May. The Dutch team achieved an outstanding result: one gold medal, three silver medals and five bronze medals. For more information about the BxMO (including the 2020 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2020 was selected by three team selection tests on 10, 11 and 12 June, each lasting four hours. Since the IMO had been postponed to September, we decided to participate in the new virtual event Cyberspace Mathematical Competition (CMC) in July with the same six students. Three young, promising students were selected to accompany the team to the IMO/CMC training camp, which was held from 4 until 11 July in Egmond aan Zee.

For younger students the Junior Mathematical Olympiad was held in October 2019 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for the virtual IMO 2020 consists of

- Jesse Fitié (18 years old)
  - bronze medal at IMO 2019
- Jovan Gerbscheid (17 years old)
  - silver medal at BxMO 2018
  - bronze medal at IMO 2018, bronze medal at IMO 2019
- Jippe Hoogeveen (17 years old)
  - bronze medal at IMO 2018, bronze medal at IMO 2019
- Rafaël Houkes (18 years old)
  - gold medal at BxMO 2020
- Tjeerd Morsch (18 years old)
  - bronze medal at BxMO 2019, silver medal at BxMO 2020
  - observer C at IMO 2019
- Hanne Snijders (18 years old)
  - bronze medal at EGMO 2019, bronze medal at EGMO 2020

Also part of the IMO/CMC selection, but not officially part of the IMO team, are:

- Jelle Bloemendaal (16 years old)
  - bronze medal at BxMO 2019, silver medal at BxMO 2020
- Kevin van Dijk (16 years old)
  - bronze medal at BxMO 2020
- Casper Madlener (15 years old)
  - silver medal at BxMO 2020

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Jeroen Huijben (observer B), University of Amsterdam

First Round, January 2019

Problems

A-problems

1. Arthur has written down five distinct positive integers smaller than 10. If you add any two of these five numbers, then the result will never be equal to 10.
Which number did Arthur write down for sure?

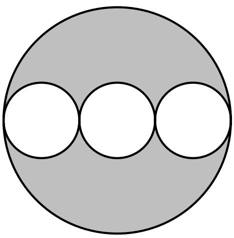
A) 1 B) 2 C) 3 D) 4 E) 5

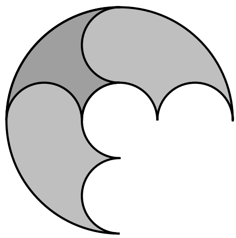
2. On a 2019×2019 chess board, there is a contagious disease. Each day some of the squares on the chess board are sick and the rest are healthy. A healthy square bordering a sick square (along a side), becomes sick itself the next day. A sick square will always be healthy the next day. A healthy square that has been sick before, can become sick again (if it is infected by one of the adjacent squares). On day 1, only the middle square is sick.
How many squares are sick on day 100?

A) 200 B) 298 C) 396 D) 9999 E) 10000

3. Out of a circular disk of radius 3, we cut three small disks of radius 1 in the way depicted in the figure on the left. This causes the remainder of the big disk to fall apart into two pieces. The bottom part is rotated 90 degrees and is put on top of the upper part as shown in the figure on the right. The part where the two pieces overlap is coloured a bit darker.
What is the total area of the figure on the right (i.e. both the light and dark grey parts together)?

A) 4π B) $\frac{9}{2}\pi$ C) $\frac{19}{4}\pi$ D) 5π E) $\frac{21}{4}\pi$





4. There are 13 distinct multiples of 7 that consist of two digits. You want to create a longest possible chain consisting of these multiples, where two multiples can only be adjacent if the last digit of the left multiple equals the first digit of the right multiple. You can use each multiple at most once. For example, 21 – 14 – 49 is an admissible chain of length 3.
What is the maximum length of an admissible chain?

A) 6 B) 7 C) 8 D) 9 E) 10

5. In a table with two rows and five columns, each of the squares is coloured black or white according to the following rules:

- Two adjacent columns may never have the same number of black squares.
- Two $2{\times}2$-squares that overlap in one column may never have the same number of black squares.

How many possible colourings of the table comply with these rules?

A) 6 B) 8 C) 12 D) 20 E) 24

6. Which of the following numbers is the largest number you can get by separating the numbers 1, 2, 3, 4, and 5 by using each of the operations $+$, $-$, $:$, and $\times$ exactly once, where you may use parentheses to indicate the order in which the operations should be executed? For example: $(5-3)\times(4+1):2=5$.

A) 21 B) $\frac{53}{2}$ C) 33 D) $\frac{69}{2}$ E) 35

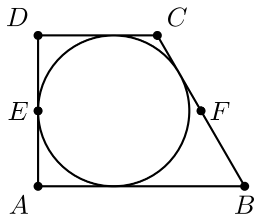
7. Agatha, Isa and Nick each have a different kind of bike. One of them has an electric bike, one has a racing bike, and one has a mountain bike. The bikes have different colours: green, blue and black. The three owners make two statements each, of which one is true and the other is false:

- Agatha says: “I have an electric bike. Isa has a blue bike.”
- Isa says: “I have a mountain bike. Nick has an electric bike.”
- Nick says: “I have a blue bike. The racing bike is black.”

Exactly one of the following statements is certainly true. Which one?

A) Agatha has a green bike.
B) Agatha has a mountain bike.
C) Isa has a green bike.
D) Isa has a mountain bike.
E) Nick has an electric bike.

8. Quadrilateral $ABCD$ has right angles at A and D. A circle of radius 10 fits neatly inside the quadrilateral and touches all four sides. The length of edge BC is 24. The midpoint of edge AD is called E and the midpoint of edge BC is called F.
What is the length of EF?



A) $\frac{43}{2}$ B) $\frac{13}{2}\sqrt{11}$ C) $\frac{33}{5}\sqrt{11}$ D) 22 E) $\frac{45}{2}$

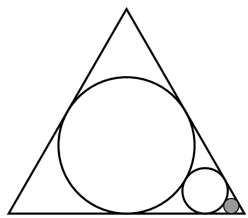
B-problems

The answer to each B-problem is a number.

1. Every day, Maurits bikes to school. He can choose between two different routes. Route B is 1.5 km longer than route A. However, because he encounters fewer traffic lights, his average speed along route B is 2 km/h higher than along route A. This makes that travelling along the two routes takes exactly the same amount of time.
How long does it take for Maurits to bike to school?

2. Starting with a positive integer, a *fragment* of that number is any positive number obtained by removing one or more digits from the beginning and/or end of that number. For example: the numbers 2, 1, 9, 20, 19, and 201 are the fragments of 2019.
What is the smallest positive integer n such that the following holds: there is a fragment of n such that when you add this fragment to n itself, you get 2019?

3. Inside an equilateral triangle, a circle is drawn that touches all three sides. The radius of the circle is 10. A second, smaller, circle touches the first circle and two sides of the triangle. A third, even smaller, circle touches the second circle and two sides of the triangle (see the figure). What is the radius of the third circle?



4. Alice has a number of cards. Each card contains three of the letters A to I. For any choice of two of those letters, there is at least one card that contains both letters.
What is the smallest number of cards that Alice can have?

Solutions

A-problems

1. E) 5

2. E) 10000

3. D) 5π

4. B) 7

5. D) 20

6. E) 35

7. D) Isa has a mountain bike.

8. D) 22

B-problems

1. 45 minutes

2. 1836

3. $\frac{10}{9}$

4. 12

Second Round, March 2019

Problems

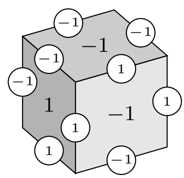
B-problems

The answer to each B-problem is a number.

B1. After breakfast, the sisters Anna and Birgit depart for school, each going to a different school. Their house is next to a bicycle path running between the two schools. Anna is cycling with a constant speed of 12 km per hour and Birgit is walking in the opposite direction with a constant speed of 4 km per hour. They depart at the same time. Shortly after their departure, mother notes that the girls have forgotten their lunch and decides to go after them. Exactly 10 minutes after Anna and Birgit have left, mother departs on her electric bike. First, she catches up with Anna. She hands her a lunch box, immediately turns around, and goes after Birgit. When she catches up with Birgit, she hands her a lunch box and immediately rides back home. Mother always rides at a constant speed of 24 km per hour.
How many minutes after the departure of Anna and Birgit does mother return home?

B2. In a tall hat there are one hundred notes, numbered from 1 to 100. You want to have three notes with the property that each of the three numbers is smaller than the sum of the other two. For example, the three notes numbered 10, 15, and 20 would be suitable (as $10 < 15 + 20$, $15 < 10 + 20$, and $20 < 10 + 15$), but the notes numbered 3, 4, and 7 would not (as 7 is not smaller than $3 + 4$). You may (without looking at the numbers on the notes) take some notes from the hat.
What is the smallest number of notes you have to take to be sure to have three notes that meet your wish?

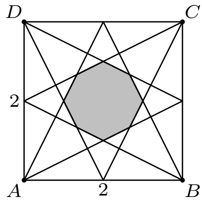
B3. On each of the twelve edges of a cube we write the number 1 or -1. For each face of the cube, we multiply the four numbers on the edges of this face and write the outcome on this face. Finally, we add the eighteen numbers that we wrote down.
What is the smallest (most negative) result we can get?



In the figure you see an example of such a cube. You cannot see the numbers on the back of the cube.

B4. If you try to divide the number 19 by 5, you will get a remainder. The number 5 fits 3 times in 19 and you will be left with 4 as remainder. There are two positive integers n having the following property: if you divide n^2 by $2n + 1$, you will get a remainder of 1000.
What are these two integers?

B5. In a square $ABCD$ of side length 2 we draw lines from each vertex to the midpoints of the two opposite sides. For example, we connect A to the midpoint of BC and to the midpoint of CD. The eight resulting lines together bound an octagon inside the square (see figure).
What is the area of this octagon?



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. We consider sequences $a_1, a_2, \ldots, a_n$ consisting of n integers. For given $k \leq n$, we can partition the numbers of the sequence into k groups as follows: a_1 goes in the first group, a_2 in the second group, and so on until a_k which goes in the k-th group. Then a_{k+1} goes in the first group again, a_{k+2} in the second group, and so on. The sequence is called k-*composite* if this partition has the property that the sums of the numbers in the k groups are equal.

The sequence $1, 2, 3, 4, -2, 6, 13, 12, 17, 8$, for instance, is 4-composite as

$$1 + (-2) + 17 \;=\; 2 + 6 + 8 \;=\; 3 + 13 \;=\; 4 + 12.$$

However, this sequence is not 3-composite, as the sums $1 + 4 + 13 + 8$, $2 + (-2) + 12$, and $3 + 6 + 17$ do not give equal outcomes.

(a) Give a sequence of 6 *distinct* integers that is both 2-composite and 3-composite.

(b) Give a sequence of 7 *distinct* integers that is 2-composite, 3-composite, and 4-composite.

(c) Find the largest $k \leq 99$ for which there exists a sequence of 99 *distinct* integers that is k-composite. (Give an example of such a sequence and prove that such a sequence does not exist for greater values of k.)

C2. A year is called *interesting* if it consists of four distinct digits. For example, the year 2019 is interesting. It is even true that all years from 2013 up to and including 2019 are interesting: a sequence of seven consecutive interesting years.

(a) Determine the next sequence of seven consecutive interesting years and prove that this is indeed the next such sequence.

(b) Prove that there is no sequence of eight consecutive interesting years within the years from 1000 to 9999.

Solutions

B-problems

1. 42

2. 11

3. -12

4. 666 and 1999

5. $\frac{2}{3}$

C-problems

C1. (a) An example of a correct sequence is $5, 7, 6, 3, 1, 2$. This sequence consists of six distinct numbers and is 2-composite since $5 + 6 + 1 = 7 + 3 + 2$. It is also 3-composite since $5 + 3 = 7 + 1 = 6 + 2$.

This is just one example out of many possible correct solutions. Below we describe how we found this solution.

We are looking for a sequence $a_1, a_2, a_3, a_4, a_5, a_6$ that is 2-composite and 3-composite. Hence, we need that

$$a_1 + a_4 = a_2 + a_5 = a_3 + a_6 \qquad \textit{and} \qquad a_1 + a_3 + a_5 = a_2 + a_4 + a_6.$$

If we choose $a_4 = -a_1$, $a_5 = -a_2$, and $a_6 = -a_3$, then the first two equations hold. The third equation gives us $a_1 + a_3 - a_2 = a_2 - a_1 - a_3$, and therefore $a_1 + a_3 = a_2$. We choose $a_1 = 1$, $a_3 = 2$ (and therefore $a_2 = 3$). We obtain the sequence $1, 3, 2, -1, -3, -2$ consisting of six distinct integers. If we wish to do so, we can increase all six numbers by 4 to get a solution with only positive numbers: $5, 7, 6, 3, 1, 2$.

(b) A possible solution is $8, 17, 26, 27, 19, 10, 1$. This sequence consists of seven distinct integers and is 2-composite since $8 + 26 + 19 + 1 = 17 + 27 + 10$. It is 3-composite since $8 + 27 + 1 = 17 + 19 = 26 + 10$. It is also 4-composite since $8 + 19 = 17 + 10 = 26 + 1 = 27$.

This is just one example out of many possible correct solutions. Below we describe how we found this solution.

We are looking for a sequence $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ *that is* 2-, 3-, *and* 4-*composite. Hence, we need that*

$$a_1 + a_3 + a_5 + a_7 = a_2 + a_4 + a_6, \tag{1}$$
$$a_1 + a_4 + a_7 = a_2 + a_5 = a_3 + a_6, \tag{2}$$
$$a_1 + a_5 = a_2 + a_6 = a_3 + a_7 = a_4. \tag{3}$$

We notice that in equation (1) we have $a_1 + a_5$ *and* $a_3 + a_7$ *on the left, and* $a_2 + a_6$ *and* a_4 *on the right. If the sequence is* 4-*composite, these four numbers are equal. Hence, we find that a* 4-*composite sequence is automatically* 2-*composite as well.*
From the equations in (3) it follows that

$$a_1 = a_4 - a_5, \qquad a_2 = a_4 - a_6, \qquad a_3 = a_4 - a_7.$$

Substituting this in the equations (2), we obtain

$$2a_4 + a_7 - a_5 = a_4 + a_5 - a_6 = a_4 + a_6 - a_7.$$

Subtracting a_4 *from each part, we get*

$$a_4 + a_7 - a_5 = a_5 - a_6 = a_6 - a_7.$$

Hence, we obtain

$$\begin{aligned} a_4 &= (a_5 - a_6) - (a_7 - a_5) = 2a_5 - a_6 - a_7, \\ a_5 &= (a_6 - a_7) + a_6 = 2a_6 - a_7. \end{aligned}$$

We have thus expressed a_1, a_2, a_3, a_4, *and* a_5 *in terms of* a_6 *and* a_7. *Every solution is obtained by a suitable choice of* a_6 *and* a_7 *for which the seven numbers become distinct. We try* $a_6 = 10$ *and* $a_7 = 1$, *and find:*

$$a_5 = 2 \cdot 10 - 1 = 19, \qquad a_4 = 2 \cdot 19 - 10 - 1 = 27, \qquad a_3 = 27 - 1 = 26,$$

$$a_2 = 27 - 10 = 17, \qquad \textit{and} \qquad a_1 = 27 - 19 = 8.$$

Hence, we have found a solution.

(c) The largest k for which a k-composite sequence of 99 distinct integers exists, is $k = 50$. An example of such a sequence is

$$1, 2, \ldots, 48, 49, 100, 99, 98, \ldots, 52, 51.$$

The 99 integers in the sequence are indeed distinct and we see that $1 + 99 = 2 + 98 = \ldots = 48 + 52 = 49 + 51 = 100$, so this sequence is 50-composite.

Now suppose that $k > 50$ and that we have a k-composite sequence $a_1, a_2, \ldots, a_{99}$. Consider the group that contains the number a_{49}. Since $49 - k < 0$ and $49 + k > 99$, this group cannot contain any other number beside a_{49}. Next, consider the group containing the number a_{50}. Since $50 - k < 0$ and $50 + k > 99$, this group cannot contain any other number beside a_{50}. Hence, the numbers a_{49} and a_{50} each form a group by themselves and must therefore have the same value. But this is not allowed since the 99 numbers in the sequence had to be distinct.

C2. (a) The years 2103 to 2109 are seven consecutive interesting years. If there is an earlier sequence of seven, then it must start before 2100. We shall now prove that this is not possible.

Because an interesting year cannot end with the digits 99, the first two digits are the same for all years in a sequence of consecutive interesting years (of four digits). Now suppose we have seven consecutive years starting with digits 20. The seven final digits are consecutive and unequal to 0 and 2, and therefore also unequal to 1. The seven final digits must be the digits 3 to 9, in this exact order. Hence, the third digit must be the only remaining digit, namely digit 1. We conclude that 2013 to 2019 is the only sequence of seven consecutive interesting years between 2000 and 2100.

(b) Suppose that there is a sequence of eight consecutive interesting years between 1000 and 9999. Because an interesting year cannot end with 99, all eight years have the same first two digits. If also the third digit does not change, then there are only 7 possibilities for the last digit, which is not enough. Therefore, there are two consecutive years in our sequence of the shape $abc9$ and $abd0$ with $d = c + 1$. Because there are eight possible final digits, these must be the eight digits unequal to a and b. Hence, both c and $d = c + 1$ must occur as final digit. Because the numbers $abcc$ and $abdd$ cannot occur, this means that in our sequence both $abcd$ and $abdc$ must occur. The difference between these two numbers is 9, and our sequence consists of eight consecutive numbers. This is also not possible. We have obtained a contradiction, and conclude that the assumption that there exists a sequence of eight consecutive interesting years between 1000 and 9999 is wrong.

Final Round, September 2019

Problems

1. A *complete number* is a 9 digit number that contains each of the digits 1 to 9 exactly once. The *difference number* of a number N is the number you get by taking the differences of consecutive digits in N and then stringing these digits together. For instance, the difference number of 25143 is equal to 3431. The complete number 124356879 has the additional property that its difference number, 12121212, consists of digits alternating between 1 and 2.
Determine all a with $3 \leq a \leq 9$ for which there exists a complete number N with the additional property that the digits of its difference number alternate between 1 and a.

2. There are n guests at a party. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is *not* friends with two other guests, those two other guests cannot be friends with each other either.
What are the possible values of n?

3. Points A, B, and C lie on a circle with centre M. The reflection of point M in the line AB lies inside triangle ABC and is the intersection of the angular bisectors of angles A and B. (The angular bisector of an angle is the line that divides the angle into two equal angles.) Line AM intersects the circle again in point D.
Show that $|CA| \cdot |CD| = |AB| \cdot |AM|$.

4. The sequence of Fibonacci numbers $F_0, F_1, F_2, \ldots$ is defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all $n \geq 0$. For example, we have

$$F_2 = F_0 + F_1 = 2, \quad F_3 = F_1 + F_2 = 3, \quad F_4 = F_2 + F_3 = 5, \quad F_5 = 8.$$

The sequence $a_0, a_1, a_2, \ldots$ is defined by

$$a_n = \frac{1}{F_n F_{n+2}} \quad \text{for all } n \geq 0.$$

Prove that for all $m \geq 0$ we have:

$$a_0 + a_1 + a_2 + \cdots + a_m < 1.$$

5. Thomas and Nils are playing a game. They have a number of cards, numbered 1, 2, 3, et cetera. At the start, all cards are lying face up on the table. They take alternate turns. The person whose turn it is, chooses a card that is still lying on the table and decides to either keep the card himself or to give it to the other player. When all cards are gone, each of them calculates the sum of the numbers on his own cards. If the difference between these two outcomes is divisible by 3, then Thomas wins. If not, then Nils wins.

(a) Suppose they are playing with 2018 cards (numbered from 1 to 2018) and that Thomas starts. Prove that Nils can play in such a way that he will win the game with certainty.

(b) Suppose they are playing with 2020 cards (numbered from 1 to 2020) and that Nils starts. Which of the two players can play in such a way that he wins with certainty?

Solutions

1. For $a = 4$, an example of such a number is 126734895. For $a = 5$, an example is the number 549832761. (There are other solutions as well.)

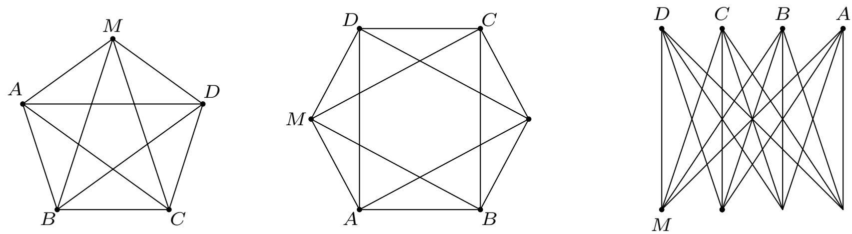
We will show that for $a = 3, 6, 7, 8, 9$ there is no complete number with a difference number equal to $1a1a1a1a$. It then immediately follows that there is also no complete number N with difference number equal to $a1a1a1a1$ (otherwise, we could write the digits of N in reverse order and obtain a complete number with difference number $1a1a1a1a$).

For a equal to 6, 7, 8, and 9, no such number N exists for the following reason. For the digits 4, 5, and 6, there is no digit that differs by a from that digit. Since the difference number of the complete number N is equal to $1a1a1a1a$, every digit of N, except the first, must be next to a digit that differs from it by a. Hence, the digits 4, 5, and 6 can only occur in the first position of N, which is impossible.

For $a = 3$ the argument is different. If we consider the digits that differ by 3, we find the triples 1–4–7, 2–5–8, and 3–6–9. If the 1 is next to the 4 in N, the 7 cannot be next to the 4 and so the 7 must be the first digit of N. If the 1 is not next to the 4, the 1 must be the first digit of N. In the same way, either the 2 or the 8 must be the first digit of N as well. This is impossible.

2. We first consider the friends of one guest, say Marieke. We know that Marieke has exactly four friends at the party, say Aad, Bob, Carla, and Demi. The other guests (if there are any other guests) are not friends with Marieke. Hence, they cannot have any friendships among themselves and can therefore only be friends with Aad, Bob, Carla, and Demi. Since everyone has exactly four friends at the party, each of them must be friends with Aad, Bob, Carla, and Demi (and with no one else).

Since Aad also has exactly four friends (including Marieke), the group of guests that are not friends with Marieke can consist of no more than three people. If the group consists of zero, one, or three people, we have the following solutions (two guests are connected by a line if they are friends):

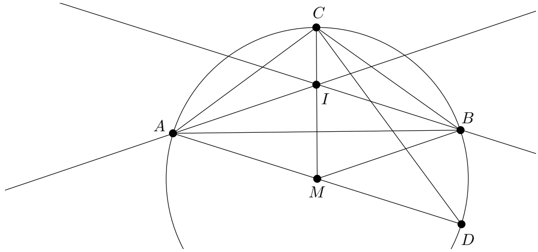


Solutions with five, six, and eight guests in total.

Now we will show that it is not possible for this group to consist of two people. In that case, Aad would have exactly one friend among Bob, Carla, and Demi. Assume, without loss of generality, that Aad and Bob are friends. In the same way, Carla must be friends with one of Aad, Bob, and Demi. Since Aad and Bob already have four friends, Carla and Demi must be friends. However, since they are both not friends with Aad, this contradicts the requirement in the problem statement.

We conclude that there can be five, six, or eight guests at the party. Hence, the possible values for n are 5, 6, and 8.

3. Let I be the reflection of point M in the line AB. We define $\alpha = \angle CAI$ and $\beta = \angle CBI$. Since AI is the angular bisector of $\angle CAB$, we find that $\angle IAB = \alpha$. Since I is the reflection of M in the line AB, we find that $\angle BAM = \alpha$. Triangle AMC is isosceles with apex M, because $|AM| = |CM|$. Hence, we see that $\angle MCA = \angle CAM = 3\alpha$. In the same way, we see that $\angle IBA = \angle ABM = \beta$ and $\angle MCB = 3\beta$. The sum of the angles of triangle ABC is therefore $2\alpha + (3\alpha + 3\beta) + 2\beta = 180°$. From this, we conclude that $\alpha + \beta = \frac{180°}{5} = 36°$, and hence that $\angle ACB = 3\alpha + 3\beta = 3 \cdot 36° = 108°$.



Since MAB is an isosceles triangle (as $|AM| = |BM|$), we see that $\alpha = \beta = 18°$. It follows from this that $\angle CAB = 2\alpha = \angle ABC$ and therefore that triangle ACB is isosceles. By considering the sum of the angles in triangle AMC, we find that $\angle AMC = 180° - 6\alpha = 72°$. Hence we also find that $\angle CMD = 180° - \angle AMC = 108°$. We have already seen that $\angle ACB = 108°$. It follows that triangles ACB and CMD are both isosceles triangles with an angle of 108° at the apex. Hence, they are similar triangles. This implies that $\frac{|CM|}{|CD|} = \frac{|AC|}{|AB|}$. By multiplying by both denominators and observing that $|CM| = |AM|$, we obtain the required result.

4. Note that for all $n \geq 0$ the number a_n can be rewritten as follows:

$$a_n = \frac{1}{F_n F_{n+2}} = \frac{F_{n+1}}{F_n F_{n+2}} \cdot \frac{1}{F_{n+1}} = \frac{F_{n+2} - F_n}{F_n F_{n+2}} \cdot \frac{1}{F_{n+1}}$$

$$= \left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right) \cdot \frac{1}{F_{n+1}} = \frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1} F_{n+2}}.$$

We now get that for each $m \geq 0$ the sum $a_0 + a_1 + a_2 + \cdots + a_m$ equals

$$\left(\frac{1}{F_0 F_1} - \frac{1}{F_1 F_2}\right) + \left(\frac{1}{F_1 F_2} - \frac{1}{F_2 F_3}\right) + \cdots + \left(\frac{1}{F_m F_{m+1}} - \frac{1}{F_{m+1} F_{m+2}}\right).$$

In this sum all terms cancel, except the first and last. In this way, we get

$$a_0 + a_1 + a_2 + \cdots + a_m = \frac{1}{F_0 F_1} - \frac{1}{F_{m+1} F_{m+2}} = 1 - \frac{1}{F_{m+1} F_{m+2}} < 1.$$

5. (a) Thomas and Nils both make 1009 moves and Nils makes the last move. Nils can make sure that the last card on the table contains a number that is *not* divisible by 3. Indeed, he could start taking cards with numbers that are divisible by 3, until all these cards are gone. Because there are only 672 such cards, he has enough turns to achieve that.

We now consider the situation before the last move of Nils. Let k be the number on the last card, and let the sums of the numbers of Thomas and Nils at that very moment be a and b. Nils has two options. If he gives away the last card, the difference between the outcomes becomes $(a+k)-b$, and if he keeps the card, the difference becomes $a-(b+k)$. Nils is able to win, unless both numbers are divisible by 3. But in that case $(a+k-b)-(a-b-k)=2k$ would also be divisible by 3. Because k is not divisible by 3, the number $2k$ is also not divisible by 3 and hence Nils can win with certainty.

(b) Nils can win. We distinguish three types of cards, depending on the number on the card: type 1 (the number has remainder 1 when dividing by 3), type 2 (the number has remainder 2 when dividing by 3), and type 3 (the number is divisible by 3). Because $2019 = 3 \cdot 673$ and the card 2020 is of type 1, there are 674 cards of type 1, 673 cards of type 2, and 673 cards of type 3.

In order to win, Nils chooses a card of type 3 in his first turn (and gives it to Thomas). Then there are 674 cards of type 1 left, 673 of type 2, and 672 of type 3. In the next turns he responds to Thomas's move in the following way (as long as he is able to).

(i) If Thomas chooses a card of type 1, then Nils chooses a card of type 2 and gives it to the same person that got Thomas's card.

(ii) If Thomas chooses a card of type 2, then Nils chooses a card of type 1 and gives it to the same person that got Thomas's card.

(iii) If Thomas chooses a card of type 3, then Nils does the same (and gives the card to Thomas).

As long as Nils keeps this up, the sum of each player's cards is divisible by 3 after his turn (because a number of type 1 and a number of type 2 add up to a number which is divisible by 3).

Because the number of cards of type 3 is always *even* after Nils's turn, Nils can always execute his planned move in case (iii). Because the number of cards of type 1 is always 1 greater than that of type 2 after Nils's turn, he can also always execute his planned move in case (ii). Only at the moment when all cards of type 2 are gone and Thomas takes the last card of type 1 (case (i)), Nils cannot execute his planned move. However, in that case Nils cannot lose anymore. Indeed, after Thomas's turn the sum of the cards of one player is still divisible by 3, but the sum of the cards of the other player is not divisible by 3 anymore. Because there are only cards of type 3 left now, this will stay the same until all cards are gone. At the end, the difference between the sums of both players is not divisible by 3 and Nils wins.

BxMO Team Selection Test, March 2020

Problems

1. For an integer $n \geq 3$ we consider a circle containing n vertices. To each vertex we assign a positive integer, and these integers do not necessarily have to be distinct. Such an assignment of integers is called *stable* if the product of any three adjacent integers is n. For how many values of n with $3 \leq n \leq 2020$ does there exist a stable assignment?

2. In an acute triangle ABC the foot of the altitude from A is called D. Let D_1 and D_2 be reflections of D in AB and AC, respectively. The intersection of BC and the line through D_1 parallel to AB, is called E_1. The intersection of BC and the line through D_2 parallel to AC, is called E_2. Prove that D_1, D_2, E_1, and E_2 lie on a circle whose centre lies on the circumcircle of $\triangle ABC$.

3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2y) + 2f(y^2) = \big(x^2 + f(y)\big) \cdot f(y)$$

for all $x, y \in \mathbb{R}$.

4. On a circle with centre M there are three distinct points A, B, and C such that $|AB| = |BC|$. The point D lies inside the circle in such a way that $\triangle BCD$ is isosceles. The second intersection point of AD and the circle is called F. Prove that $|FD| = |FM|$.

5. A set S consisting of 2019 (distinct) positive integers has the following property: the product of any 100 elements of S is a divisor of the product of the other 1919 elements. What is the maximum number of prime numbers that S could contain?

Solutions

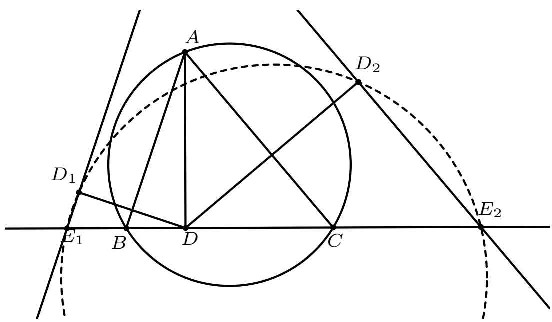
1. Suppose n is not a multiple of 3 and that we have a stable assignment of the numbers $a_1, a_2, \ldots, a_n$, in that order on the circle. Then we have $a_i a_{i+1} a_{i+2} = 3$ for all i, where the indices are considered modulo n. Hence,

$$a_{i+1}a_{i+2}a_{i+3} = n = a_i a_{i+1} a_{i+2},$$

which yields $a_{i+3} = a_i$ (as all numbers are positive). Through induction, we find that $a_{3k+1} = a_1$ or all integers $k \geq 0$. Because n is not a multiple of 3, the numbers $3k+1$ for $k \geq 0$ take on all values modulo n: indeed, 3 has a multiplicative inverse modulo n, hence $k \equiv 3^{-1} \cdot (b-1)$ implies $3k+1 \equiv b \mod n$ for all b. We conclude that all numbers on the circle must equal a_1. Hence, we have $a_1^3 = n$, where a_1 is a positive integer. Hence, if n is not a multiple of 3, then n must be a cube.

If n is a multiple of 3, then we put the numbers $1, 1, n, 1, 1, n, \ldots$ in that order on the circle. In that case, the product of three adjacent numbers always equals $1 \cdot 1 \cdot n = n$. If n is a cube, say $n = m^3$, then we put the numbers $m, m, m, \ldots$ on the circle. In that case, the product of three adjacent numbers always equals $m^3 = n$.

We conclude that a stable assignment exists if and only if n is a multiple of 3, or a cube. Now we have the count the number of such n. The multiples of 3 with $3 \leq n \leq 2020$ are $3, 6, 9, \ldots, 2019$; these are $\frac{2019}{3} = 673$ numbers. The cubes with $3 \leq n \leq 2020$ are $2^3, 3^3, \ldots, 12^3$, because $12^3 = 1728 \leq 2020$ and $13^3 = 2197 > 2020$. These are 11 cubes, of which 4 are divisible by 3, hence there are 7 cubes which are not a multiple of 3. Altogether, there are $673 + 7 = 680$ values of n satisfying the conditions. $\square$



2. Let K be the midpoint of DD_1, and let L be the midpoint of DD_2. Then K lies on AB and L lies on AC. Because $\angle AKD = 90° = \angle ALD$, the quadrilateral $AKDL$ is cyclic. Hence, $\angle DLK = \angle DAK = \angle DAB = 90^\circ - \angle ABC$. Moreover, KL is a midsegment in triangle DD_1D_2, hence $\angle DLK = \angle DD_2D_1$. We conclude that $\angle DD_2D_1 = 90^\circ - \angle ABC$.

Because $AC \perp DD_2$ en $D_2E_2 \parallel AC$, we have $\angle DD_2E_2 = 90^\circ$. Hence, $\angle D_1D_2E_2 = \angle D_1D_2D + \angle DD_2E_2 = 90^\circ - \angle ABC + 90^\circ = 180^\circ - \angle ABC$. On the other hand, as $D_1E_1 \parallel AB$, we have $\angle D_1E_1E_2 = \angle ABC$, hence we get $\angle D_1D_2E_2 = 180^\circ - \angle D_1E_1E_2$. We conclude that $D_1E_1E_2D_2$ is a cyclic quadrilateral.

Let M be the point such that AM is a diameter of the circumcircle of $\triangle ABC$. Thales' theorem yields $\angle ACM = 90^\circ$. Hence, $CM \perp AC$, which yields $CM \perp D_2E_2$ and $CM \parallel DD_2$. Moreover, L is the midpoint of DD_2 and $LC \parallel D_2E_2$, hence LC is a midsegment in triangle DD_2E_2. This means that C is the midpoint of DE_2. Because $CM \parallel DD_2$, we get that CM is also a midsegment, hence CM intersects D_2E_2 in the middle. As $CM \perp D_2E_2$, the line CM is the perpendicular bisector of D_2E_2. Analogously, we get that BM is the perpendicular bisector of D_1E_1. Hence, M is the intersection point of the perpendicular bisectors of two of the chords of the circle through D_1, D_2, E_1, and E_2. Hence, M is the centre of this circle. $\square$

3. Substituting $x = 1$ gives $f(y) + 2f(y^2) = (1 + f(y))f(y)$, hence

$$2f(y^2) = f(y)^2. \tag{4}$$

Using this, we can cancel the $2f(y^2)$ on the left hand side of the original functional equation against the $f(y)^2$ on the right hand side:

$$f(x^2y) = x^2 f(y).$$

Substituting $y = 1$ in this equations yields $f(x^2) = x^2 f(1)$, and substituting $y = -1$ yields $f(-x^2) = x^2 f(-1)$. Because x^2 takes on all non-negative numbers as value when $x \in \mathbb{R}$, we get

$$f(x) = \begin{cases} cx & \text{als } x \geq 0, \\ dx & \text{als } x < 0, \end{cases}$$

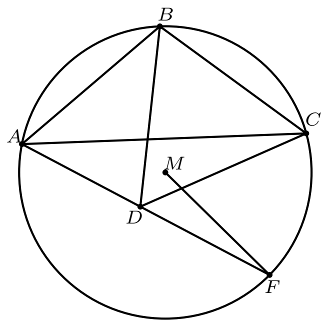
with $c = f(1)$ and $d = -f(-1)$. Now substitute $y = 1$ in equation (4), which yields $2f(1) = f(1)^2$, hence $2c = c^2$. It follows that $c = 0$ or $c = 2$. If we actually substitute $y = -1$ in equation (4), then we find that $2f(1) = f(-1)^2$, hence $2c = (-d)^2$. For $c = 0$, we get $d = 0$, and for $c = 2$, we get $d = 2$ or $d = -2$. So, there are three cases:

- $c = 0$, $d = 0$: then $f(x) = 0$ for all x;

- $c = 2$, $d = 2$: then $f(x) = 2x$ for all x;
- $c = 2$, $d = -2$: then $f(x) = 2x$ for $x \geq 0$, and $f(x) = -2x$ for $x < 0$, or, in other words, $f(x) = 2|x|$ for all x.

Using the first function, both sides of the functional equation become 0, so this function is a solution. Using the second function, we get $2x^2y + 4y^2$ on the left hand side, and $(x^2 + 2y) \cdot 2y = 2x^2y + 4y^2$ on the right hand side, so this function is a solution as well. Using the third function, we get $2|x^2y| + 4|y^2| = 2x^2|y| + 4y^2$ on the left hand side, and $(x^2 + 2|y|) \cdot 2|y| = 2x^2|y| + 4|y|^2 = 2x^2|y| + 4y^2$ on the right hand side, so also this function is a solution.

Altogether, we found the three solutions: $f(x) = 0$, $f(x) = 2x$, and $f(x) = 2|x|$. $\square$



4. We will prove that $|FD| = |FC|$ and $|FC| = |FM|$, which proves the statement.

In the cyclic quadrilateral $ABCF$, we have $\angle BCF = 180° - \angle BAF$. As $|AB| = |BC| = |BD|$ we also have $\angle BAF = \angle BAD = \angle ADB$, hence $\angle BDF = 180° - \angle ADB = 180° - \angle BAF$. We see that $\angle BCF = \angle BDF$. Moreover, $\angle DFB = \angle AFB$ and $\angle CFB$ are inscribed angles on chords AB and BC of the same length, hence $\angle DFB = \angle CFB$. Triangles BCF and BDF have two pairs of equal angles; because they also have the side BF in common, they are congruent. We conclude that $|FC| = |FD|$ and $\angle DBF = \angle CBF$.

The inscribed angle theorem yields $\angle CMF = 2\angle CBF$. From the equality $\angle DBF = \angle CBF$ we just found, we get that $2\angle CBF = \angle CBD = 60°$, hence $\angle CMF = 60°$. Moreover, $|MC| = |MF|$ (radius of the circle), hence

$\triangle CMF$ is isosceles with an angle of $60°$, which yields that the triangle is equilateral. This means that $|FC| = |FM|$.

This concludes the proof that $|FD| = |FC| = |FM|$. $\square$

5. The maximum number of prime numbers is 1819.

We start with the construction. Choose distinct primes p_1, p_2, $\ldots$, p_{1819}, and let $P = p_1 p_2 \cdots p_{1819}$. Let

$$S = \{p_1, p_2, \ldots, p_{1819}, P, P \cdot p_1, \ldots, P \cdot p_{199}\}.$$

For each p_i, there are 201 numbers in S that are divisible by p_i (namely, p_i and all multiples of P). Of these, at most one has two factors p_i; the rest has only one factor p_i. If we now take 100 numbers from S, then their product has at most 101 factors p_i. The other numbers contain at least 101 numbers which are divisible by p_i, hence their product has at least 101 factors p_i. Because this holds for any p_i, and the numbers in S do not have any other prime factors, this implies that S has the desired property.

We now prove that S cannot contain more than 1819 primes. Consider a prime divisor q of a number in S. Suppose that at most 199 numbers S are divisible by q. Then we take the 100 elements of S having the most factors q; these always have more factors q in total than the other elements, which contradicts the condition in the problem statement. Hence, there are at least 200 numbers in S which are divisible by q. If there are exactly 200, then we also get that the number of factors q in all of these numbers must be equal, otherwise we get a contradiction again by taking the 100 elements having the most factors q.

We see that S contains at least 199 non-primes, because a prime p in S divides at least 199 other elements of S. Suppose that S contains exactly 199 non-primes. Then the prime factor p in each of these 199 non-primes occurs exactly once (namely, equally often as in the prime number p). Moreover, the numbers in S cannot be divisible by a prime r that is not contained in S, because then there would be at least 200 multiples of r inside S, and these would be 200 non-primes, which is a contradiction. We get that each of the 199 non-primes in S must be the product of the primes in S. In particular, these 199 numbers are not distinct. This is a contradiction, hence S must contain at least 200 non-primes, and hence at most 1819 primes. $\square$

IMO Team Selection Test 1, June 2020

Problems

1. In an acute triangle ABC, the centre of the incircle is I, and $|AC| + |AI| = |BC|$. Prove that $\angle BAC = 2\angle ABC$.

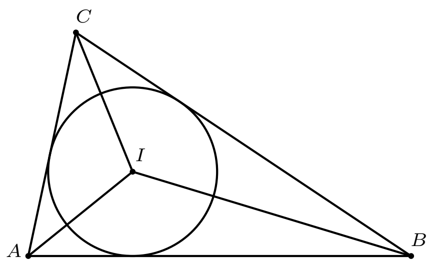
2. Determine all polynomials $P(x)$ with real coefficients for which

$$P(x^2) + 2P(x) = P(x)^2 + 2.$$

3. For a positive integer n, we consider an $n \times n$-board and tiles with sizes 1×1, 1×2, ..., $1 \times n$. In how many ways, can exactly $\frac{1}{2}n(n+1)$ squares of the board be coloured red, so that the red squares can be covered by placing the n tiles horizontally on the board, as well by placing the n vertically on the board? Two colourings which are not identical, but which can be obtained from one another by rotation or reflection, are counted as different colourings.

4. Let $a, b \geq 2$ be positive integers with $\gcd(a, b) = 1$. Let r be the smallest positive value that $\frac{a}{b} - \frac{c}{d}$ can take, where c and d are positive integers satisfying $c \leq a$ and $d \leq b$. Prove that $\frac{1}{r}$ is an integer.

Solutions



1. Let D be a point on BC such that $|CD| = |AC|$. Because $|BC| = |AC| + |AI|$, the point D lies on the interior of side BC, and we have $|BD| = |AI|$. Because triangle ACD is isosceles, the angle bisector CI is also the perpendicular bisector of AD, hence A is the reflection of D in CI. Hence, we get $\angle CDI = \angle CAI = \angle IAB$, hence $180° - \angle BDI = \angle IAB$, which means that quadrilateral $ABDI$ is cyclic. In this cyclic quadrilateral BD and AI have the same length. Therefore, AB and ID are parallel. Hence, $ABDI$ is an isosceles trapezium, which has equal angles at the base. Hence, $\angle CBA = \angle DBA = \angle BAI = \frac{1}{2}\angle BAC$, which proves the statement. $\square$

2. We rewrite the equation as

$$P(x^2) - 1 = \big(P(x) - 1\big)^2.$$

Let $Q(x) = P(x) - 1$, then Q is a polynomial with real coefficients satisfying

$$Q(x^2) = Q(x)^2.$$

Suppose that Q is constant, say $Q(x) = c$ with $c \in \mathbb{R}$. Then we have $c = c^2$, hence $c = 0$ or $c = 1$. Both possibilities give rise to solutions. Form now on, we may assume that Q is non-constant, hence we can write $Q(x) = bx^n + R(x)$ with $n \geq 1$, $b \neq 0$, and $R(x)$ a polynomial with real coefficients of degree at most $n - 1$. The polynomial equation then becomes

$$bx^{2n} + R(x^2) = b^2x^{2n} + 2bx^n \cdot R(x) + R(x)^2.$$

By comparing the coefficients in front of x^{2n} on the left and right hand side, we get $b = b^2$. As $b \neq 0$, we must have $b = 1$. Subtracting x^{2n} on both sides, yields

$$R(x^2) = 2x^n \cdot R(x) + R(x)^2.$$

If R is non-zero, then it has degree $m \geq 0$. We have $m < n$. Then the left hand side of this equation has degree $2m$, and the right hand side has degree $m + n$, as $m + n > 2m$. This is a contradiction. Hence, R must be the zero polynomial, which yields $Q(x) = x^n$. This polynomial indeed satisfies the polynomial equation for Q.

Hence, we find the following solutions: $P(x) = 1$, $P(x) = 2$, and $P(x) = x^n + 1$ with $n \geq 1$. □

3. The number of red squares must equal the total number of squares covered by the n tiles, hence the tile are only put on top of red squares. Consider a colouring of the board and the corresponding *horizontal covering* by the tiles (where all tiles are places horizontally) and the *vertical covering*. We will deduce a number of properties for the colouring, and then count how many colourings there are. The tile whose size is $1 \times k$ is called the *k-tile*.

Because the horizontal covering contains an n-tile, each column has at least one red square. In the vertical covering, each column must therefore contain at least one tile; because there are exactly n tiles, this means that there must be exactly one tile in each column. In the same way, each row must contain exactly one tile in the horizontal covering. Now number the rows and columns depending on the number of the tile that has been put there: so row i is the row containing the i-tile in the horizontal covering, and analogously for the columns.

We will now prove that the square in row i and column j (which will be called (i, j)) is red if and only if $i+j \geq n+1$. We prove this using induction on i. In row 1, there is only one red square, so that must be in the column containing the n-tile in the vertical covering, i.e. column n. Hence, the square $(1, j)$ is red if and only if $j = n$, or if and only if $1 + j \geq n + 1$. Now let $k \geq 1$ and suppose the statement has been proved for all $i \leq k$. We want to prove the statement for $i = k + 1$, i.e. that the square $(k + 1, j)$ is red if and only if $k + 1 + j \geq n + 1$, or $j \geq n - k$. Consider a column $j \geq n - k$. Because of the induction hypothesis, we know exactly how many red squares this column has in rows $1, 2, \ldots, k$: namely, the square (i, j) is red if and only if $i+j \geq n+1$, or $i \geq n+1-j$; these are $k-(n-j) = j+k-n$ squares. In the other $n - k$ rows, this column needs another $j - (j + k - n) = n - k$ red squares. Hence, this column has a red square in each of these rows, in particular in the row $i = k + 1$. In row $i = k + 1$, the squares (i, j) with $j \geq n - k$ are all red, and these are $k + 1$ squares. Hence, these are exactly all red squares in row $i = k + 1$, hence the square (i, j) is red if and only if $j \geq n - k$, or if and only if $i + j \geq n - k + k + 1 = n + 1$. This finishes the proof by induction.

Now consider two adjacent rows with row numbers a and b, with $a > b$. In column $n - b$, there is a red square row a (because $a + n - b > n$), but not in row b. In the row directly on the other side of row b (if this row exists), there cannot be a red square in column $n - b$, because the red squares in column $n - b$ would otherwise not be consecutive, and then the tile with number $n - b$ cannot lie there. The row number of this row must therefore be smaller than b. We conclude that the row numbers cannot decrease first and then increase. Above and below row n, there must be a row with a smaller number (or no row at all), and the row numbers must descend in both directions from there. We see that the row numbers from top to bottom must first ascend until we get to row n, and then they must descend. The same can be proved for the column numbers.

Vice versa, we have to prove that if the row and columns numbers are first ascending and then descending, then the horizontal and vertical tiles can be put. To prove this, we colour the square (i, j) red if and only if $i+j \geq n+1$. For a fixed i, the red squares are the squares (i, j) with $j \geq n + 1 - i$; because of the order of the column numbers, these columns are adjacent. Hence, in each row, the red squares are adjacent. The horizontal tiles can be put exactly on top of the red squares. In the same way, this can be done for the vertical tiles. For these row and column numbers, there was not other way to choose the colouring, because we already know that for each suitable colouring the square (i, j) is red if and only if $i + j \geq n + 1$.

Altogether, we are looking for the number of ways to choose the row and column numbers such that the numbers are first ascending and then descending; corresponding to each of these choices, there is exactly one way to colour the squares so that they satisfy the conditions. The number of ways to put the numbers 1 to n in an order that is first ascending and then descending, equals the number of subsets of $\{1, 2, \ldots, n-1\}$. Namely, each ordering corresponds to the subset of numbers that appear before the number n; these can be sorted in a unique way (ascending), and the rest of the numbers must be sorted descending and put after the n. The number of subsets is 2^{n-1}. Hence, the total number of colourings satisfying the conditions, is $(2^{n-1})^2 = 2^{2n-2}$. $\square$

4. We will first show that it is possible to choose c and d such that $\frac{a}{b} - \frac{c}{d} = \frac{1}{bd}$. Because $\gcd(a, b) = 1$, there exists a multiplicative inverse b^{-1} of b modulo a. Now let c with $1 \leq c \leq a$ be such that $c \equiv -b^{-1} \mod a$. Then we have $bc \equiv -1 \mod a$, hence $a \mid bc + 1$. Define $d = \frac{bc+1}{a}$, which is a positive integer. We have $d = \frac{bc+1}{a} \leq \frac{ba+1}{a} = b + \frac{1}{a}$. Because $a \geq 2$ and d is an integer, we get $d \leq b$. All conditions are met. Hence, we have $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} = \frac{bc+1-bc}{bd} = \frac{1}{bd}$.

If this is the smallest possible outcome, then we are done, because $\frac{1}{r} = bd$ would be an integer. We will show that no smaller positive outcome is achievable. Let c and d be as above, and suppose there are positive integers $c' \leq a$ and $d' \leq b$ such that $0 < \frac{a}{b} - \frac{c'}{d'} < \frac{1}{bd}$. We will derive a contradiction.

Let $x = ad' - bc'$, then we have $\frac{a}{b} - \frac{c'}{d'} = \frac{x}{bd'}$, hence $xbd < bd'$, which yields $xd < d'$. We also know that $x > 0$. Hence, $0 < xd < d' \leq b$, which means that xd and d' are two distinct numbers whose difference is less than b. Moreover, from $x = ad' - bc'$ we get that $x \equiv ad' \mod b$. On the other hand, we know that $ad - bc = 1$, hence $ad \equiv 1 \mod b$, hence $xad \equiv x \mod b$. Combining this, we get $ad' \equiv xad \mod b$. Because $\gcd(a, b) = 1$ we may divide by a, hence $d' \equiv xd \mod b$. However, we already saw that d' and xd are distinct numbers whose difference is smaller than b, hence this is impossible.

We conclude that the c and d we found indeed give the smallest possible outcome, and hence $\frac{1}{r}$ is an integer. $\square$

IMO Team Selection Test 2, June 2020

Problems

1. Let $a_1, a_2, \ldots, a_{2020}$ be real numbers, not necessarily distinct. For all $n \geq 2020$, let a_{n+1} be the minimal real root of the polynomial

$$P_n(x) = x^{2n} + a_1 x^{2n-2} + a_2 x^{2n-4} + \ldots + a_{n-1} x^2 + a_n,$$

if it exists. Assume that a_{n+1} exists for all $n \geq 2020$. Prove that $a_{n+1} \leq a_n$ for all $n \geq 2021$.

2. Ward and Gabrielle are playing a game on a large sheet of paper. At the start of the game, there are 999 ones on the sheet of paper. Ward and Gabrielle each take turns alternatingly, and Ward has the first turn. During their turn, a player must pick two numbers a and b on the sheet such that $\gcd(a, b) = 1$, erase these numbers from the sheet, and write the number $a + b$ on the sheet. The first player who is not able to do so, loses. Determine which player can always win this game.

3. Determine all pairs (a, b) of positive integers for which

$$a + b = \varphi(a) + \varphi(b) + \gcd(a, b).$$

Here, $\varphi(n)$ is the number of integers $k \in \{1, 2, \ldots, n\}$ satisfying $\gcd(n, k) = 1$.

4. Let ABC be an acute triangle and let P be the intersection of the tangents in B and C to the circumcircle of $\triangle ABC$. The line through A perpendicular to AB and the line through C perpendicular to AC intersect in a point X. The line through A perpendicular to AC and the line through B perpendicular to AB intersect in a point Y. Prove that $AP \perp XY$.

Solutions

1. If $x = \alpha$ is a root of P_n, then $x = -\alpha$ is root of P_n as well, as all terms of P_n have even degree. The minimal root of P_n therefore cannot be positive. Therefore $a_n \leq 0$ for all $n > 2020$. We have $P_{n+1}(x) = x^2 \cdot P_n(x) + a_{n+1}$. Substitute $x = a_{n+1}$; as that is a root of P_n, we have $P_{n+1}(a_{n+1}) = 0 + a_{n+1} \leq 0$.

As the maximal degree term in $P_n(x)$ is x^{2n}, there exists an $N < 0$ such that $P_n(x) > 0$ for all $x < N$. Taking for example $-N = \max(2, |a_1| + |a_2| + \ldots + |a_n|)$, we see for $x < N$ that $x^{2i-2} \leq x^{2n-2}$ for all $1 \leq i \leq n$ and therefore that

$$\begin{aligned}
&\left|a_1 x^{2n-2} + a_2 x^{2n-4} + \ldots + a_{n-1}x^2 + a_n\right| \\
&\qquad \leq \left|a_1 x^{2n-2}\right| + \left|a_2 x^{2n-4}\right| + \ldots + \left|a_{n-1}x^2\right| + |a_n| \\
&\qquad \leq |a_1| x^{2n-2} + |a_2| x^{2n-2} + \ldots + |a_{n-1}| x^{2n-2} + |a_n| x^{2n-2} \\
&\qquad \leq (|a_1| + |a_2| + \ldots + |a_n|) x^{2n-2} \\
&\qquad \leq -N \cdot x^{2n-2} \\
&\qquad < x^{2n},
\end{aligned}$$

so $x^{2n} + a_1 x^{2n-2} + a_2 x^{2n-4} + \ldots + a_{n-1}x^2 + a_n > 0$. Hence for $n \geq 2021$ there exists an $N < 0$ with $P_n(x) > 0$ for all $x < N$, whereas $P_n(a_n) \leq 0$. Therefore $P_n(x)$ has a root smaller than a_n. As a_{n+1} is the minimal root, we have $a_{n+1} \leq a_n$. $\square$

2. Gabrielle can always win using the following strategy: during each of her turns, she picks the largest two numbers on the sheet as a and b. Using induction on k, we will prove that she is always allowed to do so, and that after her k-th turn, the sheet contains the number $2k + 1$ and $998 - 2k$ ones.

In his first turn, Ward can only pick $a = b = 1$, after which the sheet contains the number 2 and 997 ones. Gabrielle then picks the two largest numbers, $a = 2$ and $b = 1$, after which the sheet contains 3 and 996 ones. This finishes the basis $k = 1$ of the induction.

Now suppose that for some $m \geq 1$ after Gabrielle's m-th turn the sheet contains the number $2m + 1$ and $998 - 2m$ ones. If $998 - 2m = 0$, then Ward cannot make a move. If not, then Ward can do one of two things, either pick $a = b = 1$ or pick $a = 2m + 1$ and $b = 1$. We consider these two cases separately:

- If Ward picks $a = b = 1$, then the sheet contains the number $2m + 1$, the number 2, and $996 - 2m$ ones. Gabrielle then picks the two largest numbers, so $a = 2m + 1$ and $b = 2$ (which is allowed since their gcd is 1). After her turn the sheet contains the numbers $2m+3 = 2(m+1)+1$ and $996 - 2m = 998 - 2(m + 1)$ ones.

- If Ward picks $a = 2m + 1$ and $b = 1$, then the sheet contains the number $2m+2$ and $997-2m$ ones. Gabrielle then picks the two largest numbers, so $a = 2m + 2$ and $b = 1$ (which is allowed since their gcd is 1). Note that there is a one left, as $997 - 2m$ is odd, so not equal to 0. After her turn the sheet contains the numbers $2m + 3 = 2(m + 1) + 1$ and $996 - 2m = 998 - 2(m + 1)$ ones.

This completes the induction.

Therefore Gabrielle can always make a move. After Gabrielle's turn 499 the only number left on the sheet is 999, so Ward can no longer make a move, and Gabrielle wins. $\square$

3. First suppose that $a = 1$. Then $\varphi(1) = 1$. For all positive integers b we have $\gcd(a, b) = 1$. Therefore in this case the equation is $1 + b = 1 + \varphi(b) + 1$, or equivalently, $\varphi(b) = b - 1$. This is equivalent to the statement that there exists a unique integer from $\{1, 2, \ldots, b\}$; which then has to be b itself (since unless $b = 1$, we have $\gcd(b, b) > 1$, but if $b = 1$ we have $\varphi(b) = b$). In other words, this is equivalent to b being a prime number. Hence the solutions for $a = 1$ are precisely the pairs $(1, p)$ with p a prime number. Similarly, the solutions for $b = 1$ are precisely the pairs $(p, 1)$ with p a prime number.

Now assume that $a, b \geq 2$. As $\gcd(b, b) > 1$ we have $\varphi(b) \leq b-1$. Therefore

$$\gcd(a, b) = a + b - \varphi(a) - \varphi(b) \geq a - \varphi(a) + 1.$$

Let p be the minimal prime divisor of a (which exists as $a \geq 2$). Since for all multiples $tp \leq a$ of p, we have $\gcd(tp, a) > 1$, it follows that $a-\varphi(a) \geq \frac{1}{p} \cdot a$. Therefore we have

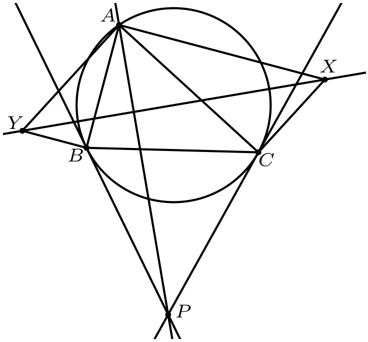
$$\gcd(a, b) \geq a - \varphi(a) + 1 \geq \frac{a}{p} + 1.$$

The two largest divisors of a are a and $\frac{a}{p}$. Since $\gcd(a, b)$ is a divisor of a that is at least $\frac{a}{p} + 1$, it must equal a. Hence $\gcd(a, b) = a$. In the same way we prove that $\gcd(a, b) = b$. So $a = b$.

The equation now is equivalent to $2a = 2\varphi(a) + a$, so also to $a = 2\varphi(a)$. Note that $2 \mid a$. Therefore write $a = 2^k \cdot m$ with $k \geq 1$ and m odd. By a well-known property of the φ-function, we have $\varphi(a) = \varphi(2^k) \cdot \varphi(m) = 2^{k-1} \cdot \varphi(m)$, and the equation becomes $2^k \cdot m = 2 \cdot 2^{k-1} \cdot \varphi(m)$, or equivalently

$m = \varphi(m)$. Therefore $m = 1$, and $a = b = 2^k$. Indeed, the equation holds for all pairs $(2^k, 2^k)$ with $k \geq 1$.

Therefore the solutions of the equation are: the pairs $(1, p)$ and $(p, 1)$ for prime numbers p, and $(2^k, 2^k)$ for all positive integers k. $\square$



4. Let M the circumcentre of $\triangle ABC$ and let $\alpha = \angle BAC$. We first show that $\triangle BYA \sim \triangle BMP$ and then that $\triangle YBM \sim \triangle ABP$. By the inscribed angle theorem we have $\angle BMC = 2\angle BAC = 2\alpha$. Quadrilateral $PBMC$ is a kite with axis of symmetry PM (by the equality of radii $|MB| = |MC|$ and equality of tangent segments $|PB| = |PC|$), so MP bisects angle $\angle BMC$. Therefore $\angle BMP = \frac{1}{2}\angle BMC = \alpha$. Moreover, we have $\angle PBM = 90°$ (tangent to a circle is perpendicular to its radius), so by the sum of angles of a triangle we have $\angle MPB = 90° - \alpha$.

On the other hand, we are given that $\angle ABY = 90°$ and we also have $\angle YAB = \angle YAC - \angle BAC = 90° - \alpha$. Therefore $\angle ABY = \angle PBM$ and $\angle YAB = \angle MPB$, from which follows that $\triangle BYA \sim \triangle BMP$. From this similarity it follows that $\frac{|YB|}{|AB|} = \frac{|MB|}{|PB|}$. Combining this with the equality of angles

$$\angle YBM = \angle YBA + \angle ABM = 90° + \angle ABM = \angle ABM + \angle MBP = \angle ABP,$$

we see that $\triangle YBM \sim \triangle ABP$.

Let T now be the intersection of AP and YM, then we have

$$\angle BYT = \angle BYM = \angle BAP = \angle BAT,$$

from which it follows that $BYAT$ is a cyclic quadrilateral. Therefore $\angle ATY = \angle ABY = 90°$, so $AT \perp YM$. Analogously, $AP \perp XM$. But from this it now follows that YM and XM coincide and we get that $AP \perp XY$. $\square$

IMO Team Selection Test 3, June 2020

1. For a positive integer n, let $d(n)$ be the number of positive divisors of n. Determine the positive integers k for which there exist positive integers a and b satisfying
$$k = d(a) = d(b) = d(2a + 3b).$$

2. Let a triangle ABC such that $|AC| < |AB|$ be given, together with its circumcircle. Let D be a varying point on the short arc AC. Let E be the reflection of A in the internal angular bisector of $\angle BDC$. Prove that the line DE passes through a fixed point, independent of where D lies.

3. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f\big(-f(x) - f(y)\big) = 1 - x - y$$
for all $x, y \in \mathbb{Z}$.

4. Suppose k and n are positive integers such that $k \leq n \leq 2k - 1$. Julian has a large pile of rectangular $k \times 1$-tiles. Merlijn picks a positive integer m, and receives from Julian m tiles to place on an $n \times n$-board. On each tile, Julian writes whether this tile should be placed horizontally or vertically. Tiles may not overlap on the board, and they must fit entirely inside the board. What is the largest number m that Merlijn can pick while still guaranteeing he can put all tiles on the board according to Julian's instructions?

Solutions

1. For $i \geq 0$, let $a = 2 \cdot 5^i$ and $b = 3 \cdot 5^i$. Then both a and b have $2(i+1)$ divisors. Moreover, we have $2a + 3b = 4 \cdot 5^i + 9 \cdot 5^i = 13 \cdot 5^i$, which also has $2(i+1)$ divisors. Therefore all even values of k satisfy the condition in the problem.

Now suppose that k is odd. Then a has an odd number of divisors and therefore is a square, say $a = x^2$. The same reasoning shows that b is also a square, say $b = y^2$, and $2a+3b$ is also a square, say $2a+3b = z^2$. Therefore we have

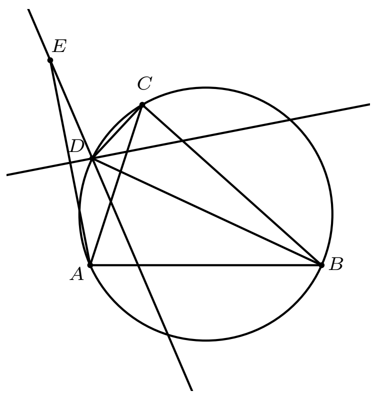
$$2x^2 + 3y^2 = z^2.$$

We show that this equation has no positive integer solutions.

Suppose for a contradiction that this equation does have a positive integer solution. Let $(x, y, z) = (u, v, w)$ be the solution with minimal $x+y+z$. So $2u^2 + 3v^2 = w^2$. Modulo 3 this equation is $2u^2 \equiv w^2$. If u is not divisible by 3, then $u^2 \equiv 1 \mod 3$, so $w^2 = 2u^2 \equiv 2 \mod 3$, however, that isn't possible. Therefore u is divisible by 3, from which it follows that w is divisible by 3 as well. Now $2u^2$ and w^2 are both divisible by 9, so $3v^2$ is divisible by 9. It follows that v is divisible by 3 as well. However, now $(x, y, z) = (\frac{u}{3}, \frac{v}{3}, \frac{w}{3})$ also satisfies the equation, and it is a solution with smaller $x + y + z$ than the supposed minimal one. This is a contradiction. Therefore the equation $2x^2 + 3y^2 = z^2$ has no positive integer solutions.

It follows that no odd k satisfies the condition in the problem. The positive integers k that satisfy that condition are therefore the even integers. □

2. Let M be the intersection of the internal angular bisector of $\angle BDC$ with the circumcircle of $\triangle ABC$. As D lies on the short arc AC, we see that M lies on the arc BC not containing A. We have $\angle BDM = \angle MDC$ as DM is the internal angular bisector of $\angle BDC$, so arcs BM and CM have the equal lengths. Hence M is independent of D.



Let S be the intersection of DE and the circumcircle of $\triangle ABC$. We show that S is independent of D. As S and M lie on the circumcircle of $\triangle ABC$, we have $\angle AMD = \angle ASD = \angle ASE$. As E is the reflection of A in DM, we now see that $\angle AME = 2\angle AMD = 2\angle ASE$.

Consider the circle with centre M passing through A. As E is the reflection of A in DM, we have $|MA| = |ME|$, so this circle also passes through E. By the inscribed angle theorem, from $\angle AME = 2\angle ASE$ it follows that S is also on this circle. Therefore S is the second intersection point of the circumcircle of $\triangle ABC$ and the circle with centre M passing through A. This is a description of S independent of D. As DE passes through S, the point S is the point required. □

3. Substituting $x = y = 1$ yields $f(-2f(1)) = -1$. Substituting $x = n$ and $y = 1$ yields $f(-f(n) - f(1)) = -n$. Substituting $x = -f(n) - f(1)$ and $y = -2f(1)$ then yields

$$f\big(-f(-f(n) - f(1)) - f(-2f(1))\big) = 1 - (-f(n) - f(1)) - (-2f(1))$$

in which the left hand side expands as $f(-(-n) - (-1)) = f(n+1)$ and the right hand as $1 + f(n) + f(1) + 2f(1) = f(n) + 3f(1) + 1$. Writing $c = 3f(1) + 1$, we then get $f(n+1) = f(n) + c$.

Applying induction in both directions, we see that $f(n+k) = f(n) + ck$ for all $k \in \mathbb{Z}$. Substituting $n = 0$ then yields $f(k) = f(0) + ck$ for all $k \in \mathbb{Z}$, so f is a linear function.

Now let $a, b \in \mathbb{Z}$ be such that $f(x) = ax + b$ for all $x \in \mathbb{Z}$. Then the left hand side of the functional equation evaluates as

$$f\big(-f(x) - f(y)\big) = a(-ax - b - ay - b) + b = -a^2x - a^2y - 2ab + b.$$

For all x and y this must be equal to $1 - x - y$. Therefore the coefficient for x on both sides must be equal (for fixed y both sides must give the same function in x), so $-a^2 = -1$, and therefore $a = 1$ or $a = -1$. If $a = -1$, substituting $x = y = 0$ yields $2b + b = 1$, which contradicts b being an integer. If $a = 1$, substituting $x = y = 0$ yields $-2b + b = 1$, so $b = -1$. Indeed, if $a = 1$ and $b = -1$, then the left hand side also expands to $1 - x - y$. Therefore the only solution to the functional equation is the function $f(x) = x - 1$. □

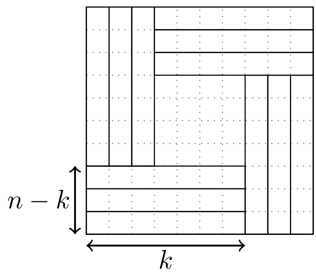
4. We show that the largest m Merlijn can pick is $\min(n, 3(n-k)+1)$. First we show that $m \leq \min(n, 3(n-k)+1)$. If Merlijn asks for $n+1$ tiles, Julian can instruct Merlijn to place them all horizontally. As $n \leq 2k-1$, it is impossible to place more than one tile horizontally on a single row, so Merlijn would need at least $n+1$ rows, this is a contradiction. Therefore $m \leq n$.

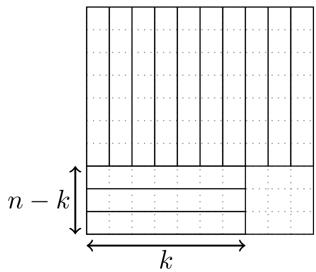
Now suppose that Merlijn asks for $3(n-k)+2$ tiles. Julian can now instruct Merlijn to place $n-k+1$ tiles vertically and $2n-2k+1$ tiles horizontally.

Note that vertical tiles always cover the $k-(n-k)=2k-n\geq 1$ rows in the middle of the board. Therefore the vertical tiles together cover at least $n-k+1$ squares in each of these rows in the middle of the board, leaving at most $k-1$ squares for the horizontal tiles. Therefore no horizontal tiles fit in these middle rows, and the $2n-2k+1$ horizontal tiles need to fit in the remaining $n-(2k-n)=2n-2k$ rows. This is a contradiction. Therefore we must have $m\leq 3(n-k)+1$. It follows that $m\leq \min(n,3(n-k)+1)$.

Now we show that it is always possible to place a pile of $\min(n,3(n+k)+1)$ tiles on the board. We first consider two special configurations. Put $n-k$ horizontal tiles into a $(n-k)\times k$-rectangle at the bottom left. To the right of that, we can fit $n-k$ more vertical tiles into a $k\times(n-k)$-rectangle in the bottom right. Above that rectangle, we can fit $n-k$ more horizontal tiles into a $(n-k)\times k$-rectangle in the top right. Finally, to the left of that, we can fit $n-k$ more vertical tiles into a $k\times(n-k)$-rectangle in the top left. In this way, we can fit $2(n-k)$ horizontal and $2(n-k)$ vertical tiles on the board.

One other way to cover the board is as follows. Place n vertical tiles in the top left, covering a $k\times n$-rectangle. Below that there is room for $n-k$ horizontal tiles to fit in an $(n-k)\times k$-rectangle. In this way, we can fit n vertical and $n-k$ horizontal tiles on the board.





Suppose that Merlijn receives A horizontal tiles and B vertical tiles from Julian. Then we have $A+B\leq n$ and $A+B\leq 3(n-k)+1$. Without loss of generality we assume that $A\leq B$. If $A\leq n-k$, then Merlijn uses the second special configuration, omitting tiles he doesn't have. As $B\leq n$, this works. Else, $A\geq n-k+1$, so $B\leq 3(n-k)+1-A\leq 3(n-k)+1-(n-k+1)=2(n-k)$. As $A\leq B$, we also have $A\leq 2(n-k)$, and Merlijn can use the first configuration, omitting tiles he doesn't have. Therefore it is always possible for Merlijn to place $\min(n,3(n-k)+1)$ tiles on the board. $\square$

Junior Mathematical Olympiad, September 2019

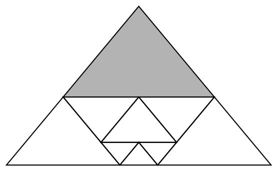
Problems

Part 1

1. At a conference, there were participants from four countries: the Netherlands, Belgium, Germany, and France. There were three times as many participants from the Netherlands as there were Belgians, and three times as many Germans as French. Five of the participants counted the total number of participants (including themselves). They counted 366, 367, 368, 369, and 370 participants, respectively. Only one of them got the right answer.
What is the the correct number of participants?

A) 366 B) 367 C) 368 D) 369 E) 370

2. We completely cover a big isosceles triangle with triangles that are similar to the big triangle as in the figure on the right.
What part of the area of the big triangle is covered by the top triangle (indicated in grey)?



A) $\frac{1}{4}$ B) $\frac{2}{7}$ C) $\frac{5}{16}$ D) $\frac{16}{49}$ E) $\frac{1}{3}$

3. In the puzzle below, a, b, c, d, and e are nonzero digits such that the two calculations are correct. The digits need not be distinct.
How many solutions are there for which $a < b$?

$$ab \times ab = cde \qquad \text{and} \qquad ba \times ba = edc.$$

A) 1 B) 2 C) 3 D) 4 E) 5

4. We say that a number is a *child* of another number if we can get it by placing between any two digits of the other number either nothing, a $+$, or a $\times$. For example, 145 and 5 are children of 12121 because $145 = 12 \times 12 + 1$ and $5 = 1 + 2 \times 1 + 2 \times 1$. The number 15 is both a child of 12121 and of 33333, because $12 + 1 + 2 \times 1 = 15 = 3 + 3 + 3 + 3 + 3$.
Which of the following numbers is also a child of both 12121 and 33333?

A) 18 B) 34 C) 39 D) 42 E) 45

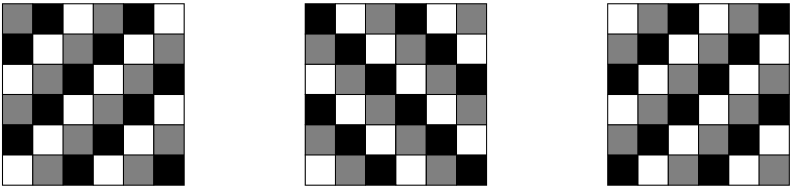
5. In a class, 30 students did a test. Every student got a grade that was an integer from 1 to 10. The grade 8 was given more often than any of the other grades.
What is the smallest possible average grade of the students?

A) $3\frac{8}{15}$ B) $3\frac{2}{3}$ C) $3\frac{5}{6}$ D) $4\frac{11}{30}$ E) $4\frac{8}{15}$

6. We want to colour the 36 squares of a 6×6 board. Every square must be coloured white, grey, or black, and the following requirement must be met:

> Three adjacent squares in the same row or column, must always have three different colours.

We say that two colourings are *truly different* if you cannot get one from the other by rotating the board. Below, you can see three colourings that meet the requirement. The first and second colouring are truly different, but the third is the same as the second after rotating.



How many truly different colourings meet the requirement (including the two from the figure)?

A) 2 B) 3 C) 4 D) 6 E) 12

7. Point D lies on side BC of triangle ABC. Angle A in triangle ABD is equal to angle C in triangle ABC, and angle A in triangle ACD is equal to angle B in triangle ABC.

The given information is not enough to derive the exact shape of triangle ABC. However, you can still derive that one of the given statements below is *always false*. Which statement is it?
By $|AB|$ we denote the length of line segment AB.

A) $|AD| < |AC|$
B) $|AC| < |AB|$
C) $|AB| < |BC|$
D) $|AD| \times |CD| < |AB| \times |AC|$
E) $|AB| \times |AC| < |AD| \times |BC|$

8. Five smart students are sitting in a circle. The teacher gives one or more marbles to each of them. He explains that he has handed out a total of

18 marbles, and that everyone got a different number of marbles. Each student is allowed to see his own number of marbles, as well as the number of marbles of his neighbour on the left and his neighbour on the right.

Using only this information, each student must try to logically deduce the difference between the numbers of marbles of the two students opposite to him. The teacher has distributed the marbles in such a way as to minimise the number of students that are able to do this. How many students can do it?

A) 0 B) 1 C) 2 D) 3 E) 5

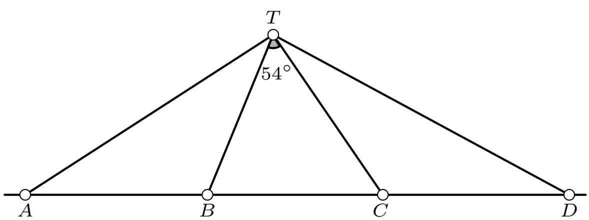
Part 2

1. We compute the square of each of the numbers from 1 to 2019. We take the last digit from each of the resulting squares, and then we add those 2019 digits together.
What number do we get?

2. The numbers *abcd* and *dcba* consist of the same four digits a, b, c, and d, but in opposite orders. When we add the two numbers, we get 13552.
Determine $a + b + c + d$.

3. One hundred students wear shirts numbered from 1 to 100. The students are arranged in a square of ten rows by ten columns. It turns out that adding the ten shirt numbers of the students in any row or any column always yields the same outcome.
Determine that outcome.

4. The four points A, B, C, and D lie on a common line (in this order). There is a point T not on the line such that $|AB| = |BT|$ and $|CD| = |CT|$. Also, angle T of triangle BTC is 54 degrees. In the figure you can see a sketch of the situation; angles and sizes are not necessarily accurate.
Determine angle T of triangle ATD.



5. Mieke has a stack of 21 cards. Mieke repeats the following operation:

> She takes the top two cards from the stack, changes their order, and then puts them at the bottom of the stack (so the top card becomes the bottom card).

Mieke repeats this operation until the cards are back in their original order. How many times does Mieke perform the operation?

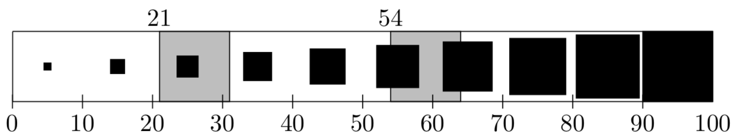
6. The number

$$2222\underbrace{9393\cdots93}_{100\text{ times }93}919$$

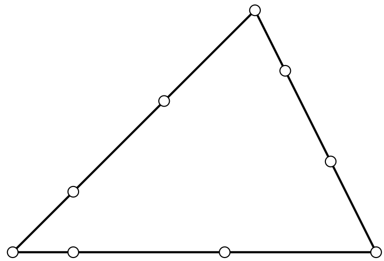
is divided by 2019.
Determine the sum of the digits of the resulting number.

7. On a white strip that is 100 mm long and 10 mm wide, ten black squares are drawn that, from left to right, have sides of length $1, 2, \ldots, 10$ mm. The centre of each black square is in the middle of the strip and $5, 15, \ldots, 95$ mm from the start (left edge) of the strip.

A transparent square is moving along the strip from left to right (indicated in grey). In the figure, two possible positions of the transparent square are depicted: in the first, its left edge is 21 mm from the start of the strip, and in the second, it is 54 mm from the start of the strip. In both cases, less than half of the part underneath the square is coloured black (only 9% in the first case). There is one position in which exactly half of the part of the strip underneath the square is coloured black.
Determine, for that position, how far the left edge of the square is from the start of the strip.



8. We are given a triangle with an additional two points on each side. So in total, there are nine points (see figure).



We want to choose three of the nine points that are not on one line. For example, we could choose (1) the three vertices of the triangle, or (2) the left vertex and the two additional points on the opposite side.
How many possible choices are there in total, including the two examples given?

Answers

Part 1

1.	C) 368	**5.**	B) $3\frac{2}{3}$
2.	D) $\frac{16}{49}$	**6.**	B) 3
3.	B) 2	**7.**	E) $\lvert AB\rvert \times \lvert AC\rvert < \lvert AD\rvert \times \lvert BC\rvert$
4.	D) 42	**8.**	D) 3

Part 2

1.	9090	**5.**	110
2.	26	**6.**	103
3.	505	**7.**	$61\frac{1}{8}$ mm
4.	117 graden	**8.**	72

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