55th Dutch Mathematical Olympiad 2016

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2017 started with the first round in January 2016, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In this first round 11101 students from 347 secondary schools participated.

The 1017 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 122 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 55th edition 2016.

The 33 most outstanding candidates of the Dutch Mathematical Olympiad 2016 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

In February a team of four girls was chosen from the training group to represent the Netherlands at the EGMO in Zürich, Switzerland, from 6 until 12 April. The team brought home a silver medal, a bronze medal, and a honourable mention; a very nice achievement. For more information about the EGMO (including the 2017 paper), see www.egmo.org.

In March a selection test of three and a half hours was held to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Namur, Belgium, from 5 until 7 May. The Dutch team received two gold medals, three silver medals and four bronze medals, and managed to get the highest total score. For more information about the BxMO (including the 2017 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2017 was selected by three team selection tests on 1, 2 and 3 June 2017, each lasting four hours. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Rio de Janeiro, from 8 until 15 July.

For younger students the Junior Mathematical Olympiad was held in October 2016 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2017 in Brazil consists of

- Nils van de Berg (17 years old)
  - bronze medal at BxMO 2017
- Wietze Koops (16 years old)
  - bronze medal at BxMO 2016, gold medal at BxMO 2017
  - honourable mention at IMO 2016
- Matthijs van der Poel (16 years old)
  - bronze medal at BxMO 2016, bronze medal at BxMO 2017
  - observer C at IMO 2016
- Levi van de Pol (15 years old)
  - silver medal at BxMO 2015, silver medal at BxMO 2016
  - observer C at IMO 2015, bronze medal at IMO 2016
- Ward van der Schoot (18 years old)
  - bronze medal at BxMO 2017
- Gabriel Visser (19 years old)
  - bronze medal at BxMO 2016
  - bronze medal at IMO 2016

We bring as observer C the promising young student

- Lammert Westerdijk (16 years old)
  - participated in BxMO 2017

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Jetze Zoethout (observer B), Utrecht University

First Round, January 2016

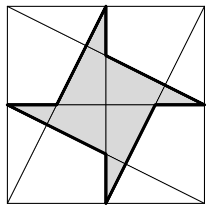
Problems

A-problems

1. Frank has two integers that add up to 26. Kees adds two more integers to it and gets 41. Pieter adds another two integers and gets 58.
At least how many of the six integers that were added up are *even*?

A) 0 B) 1 C) 2 D) 3 E) 4

2. In a square with side length 12, line segments are drawn between the vertices and the midpoints of the sides and between the midpoints of opposite sides (see the figure). In this way, a star shaped figure is created.
What is the area of this figure?

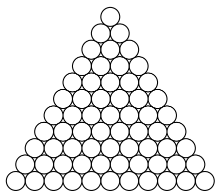


A) 12 B) 16 C) 20 D) 36 E) 48

3. A positive integer is called *fully divisible* if it is divisible by each of its digits. Moreover, these digits must all be distinct (and nonzero). For example, 162 is fully divisible, because it is divisible by 1, 6, and 2.
How many fully divisible two-digit integers are there?

A) 4 B) 5 C) 6 D) 7 E) 8

4. An *eight* is a figure consisting of two equal circles touching each other, like 8, ∞ or 8. In the figure you see 66 circles stacked in the shape of a triangle.
How many eights can you find in this stack?

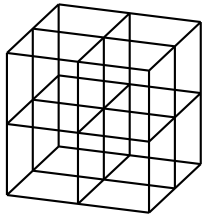


A) 99 B) 108 C) 120 D) 135 E) 165

5. Five integers are written around a circle. Two neighbouring numbers never add up to a multiple of three. Also, a number and its two neighbours never add up to a multiple of three.
How many of the five integers are multiples of three?

A) 1 B) 2 C) 3 D) 4 E) 2 and 3 are both possible

6. In the figure you see a wire-frame model of a $2{\times}2{\times}2$-cube consisting of 8 small cubes with side length 1 dm. This figure uses 54 dm of wire.
How many dm of wire are needed for a wire-frame model of a $10 \times 10 \times 10$-cube consisting of one thousand small cubes with side length 1 dm?



A) 121 B) 1000 C) 1210 D) 3000 E) 3630

7. A square board is divided into 4×4 squares. At the start, all squares are white. Now, we want to colour some of the squares blue, in such a way that each blue square will be adjacent to exactly one white square (two squares are called adjacent if they have a side in common).
What is the maximum number of squares that we can colour blue?

A) 6 B) 8 C) 10 D) 12 E) 14

8. For three *distinct* positive integers a, b, and c we have $a + 2b + 3c < 12$.
Which of the following inequalities is certainly satisfied?

A) $3a + 2b + c < 17$ B) $a + b + c < 7$ C) $a - b + c < 4$
D) $b + c - a < 3$ E) $3b + 3c - a < 6$

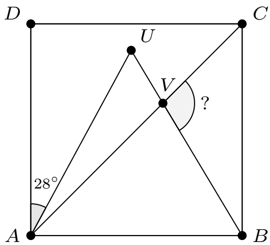
B-problems

The answer to each B-problem is a number.

1. We construct a list of all positive integers that divide 707070. The numbers are listed in decreasing order. The first number in the list is therefore 707070 and the last one is 1.
What is the seventh number in the list?

2. In the AO-language all words consist of only A's and O's and every possible sequence of A's and O's is a word. There are, for example, eight three letter words: 'OOO', 'OOA', 'OAO',. . . , 'AAO', and 'AAA'. Words that contain the letter combinations 'AO' and 'OA' equally often are called *special*. For example, 'AOAAOOOAA' is special, because the word contains both letter combinations 'AO' and 'OA' twice.
Find a special word consisting of four A's and four O's with the additional property that after removing any of its letters, the resulting seven letter word is again special.

3. In the square $ABCD$ lies a point U such that BU and AB have the same length. Point V is the intersection of BU and the diagonal AC. The size of angle DAU is 28 degrees.
What is the size of the angle at V in triangle BVC?



4. Seven people are suspects of a theft:
- **Alex**, a brown-haired man with blue eyes;
- **Boris**, a blond man with green eyes;
- **Chris**, a blond man with brown eyes;
- **Denise**, a blond woman with brown eyes;
- **Eva**, a brown-haired woman with blue eyes;
- **Felix**, a brown-haired man with brown eyes;
- **Gaby**, a blond woman with blue eyes.

Detectives Helga, Ingrid, and Julius know that one of the suspects is the thief. After conducting some investigations they share their information.

Helga: "I know the eye and hair colour of the thief, but I do not know who the thief is."

Ingrid did not hear Helga and says:
"I know the hair colour and the gender, but I do not know who the thief is."

At last, Julius says:
"First I knew only the gender of the thief, but after hearing you I know who the thief is."

The detectives spoke the truth. Who is the thief?

Solutions

A-problems

1. C) 2

2. D) 36

3. B) 5

4. E) 165

5. B) 2

6. E) 3630

7. D) 12

8. D) $b + c - a < 3$

B-problems

1. 70707

2. 'AAOOOOAA' (or 'OOAAAAOO')

3. 101°

4. Denise

Second Round, March 2016

Problems

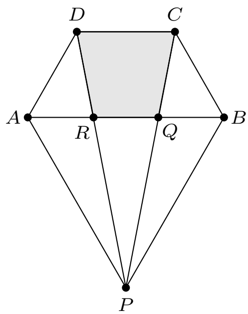
B-problems

The answer to each B-problem is a number.

B1. How many of the integers from 10 to 99 have the property that the number equals four times the sum of its two digits?

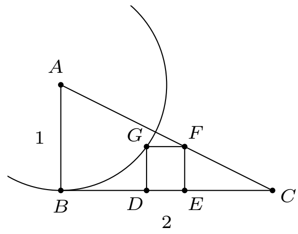
B2. In a box there are 100 cards that are numbered from 1 to 100. The numbers are written on the cards. While being blindfolded, Lisa is going to draw one or more cards from the box. After that, she will multiply together the numbers on these cards.
Lisa wants the outcome of the multiplication to be divisible by 6. How many cards does she need to draw to make sure that this will happen?

B3. In the trapezium $ABCD$ the sides AB and CD are parallel and we have $|BC| = |CD| = |DA| = \frac{1}{2}|AB|$. On the exterior of side AB there is an equilateral triangle BAP. The point Q is the intersection of PC and AB, and R is the intersection of PD and AB (see the figure). The area of triangle BAP is 12.
Determine the area of quadrilateral $QCDR$.



B4. At championships of 'The Settlers of Catan', three participants play against each other in each game. At a certain championship, three of the participants were girls and they played against each other in the first game. Each pair of participants met each other in exactly one game and in each game at least one girl was playing.
What is the maximum number of participants that could have competed in the championship?

B5. Triangle ABC has a right angle at B. Moreover, the side length of AB is 1 and the side length of BC is 2. On the side BC there are two points D and E such that E lies between C and D and $DEFG$ is a square, where F lies on AC and G lies on the circle through B with centre A. Determine the length of DE.



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. A positive integer is called 2016-*invariant* if the sum of its digits does not change when you add 2016 to the integer. For example, the integer 8312 is 2016-invariant: the sum of the digits of 8312 is $8 + 3 + 1 + 2 = 14$, and this equals the sum of the digits of $8312 + 2016 = 10328$, which is $1 + 0 + 3 + 2 + 8 = 14$.

(a) Determine the largest four-digit number that is 2016-invariant.

(b) There are 9999 positive integers having at most four digits.
Determine how many of these are 2016-invariant.

C2. For the upcoming exam, the desks in a hall are arranged in n rows containing m desks each. We know that $m \geq 3$ and $n \geq 3$. Each of the desks is occupied by a student. Students who are seated directly next to each other, in front of each other, or diagonally from each other, are called *neighbours*. Thus, students in the middle of the hall have 8 neighbours. Before the start of the exam, each student shakes hands once with each of their neighbours. In total, there are 1020 handshakes.
Determine the number of students.

Solutions

B-problems

1. 1

2. 68

3. 5

4. 7

5. $\frac{2}{5}$

C-problems

C1. When adding two integers, we write one on top of the other and then add the digits from right to left. If at a certain position the sum of the two digits is greater than 9, a *carry* occurs at this position: we must *carry a* 1 to the addition of the two digits in the next position (the position to the left).

Let n be a number whose sum of digits equals s. Now consider the number $n + 2016$. We can derive the sum of its digits from the addition procedure. If no carry occurs, then the sum of the digits of $n + 2016$ equals $s + 2 + 1 + 6 = s + 9$. If carries do occur, then with each carry the sum of the digits decreases by 9. After all, if two digits x and y add up to a number greater than 9, the digit below these two will be $x + y - 10$ instead of $x + y$, while we carry a 1 causing the digit to the left to increase by 1. In total, the sum of the digits thus decreases by 9 for each carry.

For example, taking $n = 1015$ *(and* $s = 1 + 0 + 1 + 5 = 7$*), then there will be one carry in the addition* $n + 2016 = 3031$*. For the sum of the digits we have* $3 + 0 + 3 + 1 = (s + 9) - \mathbf{1} \cdot 9$*.*
As another example, take $n = 3084$ *(where* $s = 15$*). Then, there will be two carries in the addition* $n + 2016 = 5100$ *and indeed we have* $5 + 1 + 0 + 0 = (s + 9) - \mathbf{2} \cdot 9$*.*

Hence, a number is 2016-invariant if and only if exactly one carry occurs when adding 2016 to it.

(a) After all observations in the previous paragraph, we conclude that we are looking for the greatest four-digit number having the property that you need to carry exactly one 1 when adding 2016 to it. This number is 9983. In the addition $9983 + 2016$ exactly one carry occurs and if you raise one of the digits 8 or 3, a second carry occurs.

(b) To count how many of the numbers n from 1 to 9999 give exactly one carry when adding 2016 to it, we consider four cases according to the position in which the carry occurs. In each case we consider the possible values of the digits of n.

- First, consider the case that the only carry occurs in the thousands place. The thousands digit must then be either 8 or 9. The unit digit can only be 0, 1, 2, or 3. The tens digit can be any digit except for 9 (because $1 + 9$ would give a carry in the tens place). The hundreds digit can be any value. Hence, in total there are $2 \cdot 10 \cdot 9 \cdot 4 = 720$ numbers in this case.
- There are no numbers for which a carry occurs only in the hundreds place (a carry there can only occur if we also carry a 1 from the tens place).
- Next, consider the case that the only carry occurs in the tens place. For the units digit, the only possible values are 0, 1, 2, and 3. The tens digit must be a 9. Because you now need to carry a 1 from the tens place to the hundreds place, the digit in the hundreds place cannot be a 9 anymore since this would cause another carry (for example, in the addition $990+2016$). The digit in the thousands place can take all values except 8 and 9. Hence, in total there are $8 \cdot 9 \cdot 1 \cdot 4 = 288$ numbers in this case.
- Finally, consider the case that the only carry occurs in the units place. For this digit, the possible values are 4, 5, 6, 7, 8, and 9. The digit in the tens place cannot be 9 or 8, because then another carry would occur (for example in $84 + 2016$). The hundreds digit can be any value and the thousands digit can be any value except 8 and 9. Hence, in total we find $8 \cdot 10 \cdot 8 \cdot 6 = 3840$ numbers in this case.

Altogether, there are $720 + 288 + 3840 = 4848$ numbers from 1 to 9999 that are 2016-invariant.

C2. There are four students seated in the corners, since $n, m \geq 2$. They each shook the hand of three other people. Not counting the students in the corners, there are $m-2$ students who are seated completely in the front, $m-2$ who are seated in the back, $n-2$ students who are seated in the leftmost row and $n-2$ in the rightmost row. These $2n+2m-8$ students each shook the hand of five other people. The remaining $mn-4-(2n+2m-8) = mn-2n-2m+4$ students each shook the hand of eight other people. Adding the numbers of handshakes for each of the students gives a total of

$$4 \cdot 3 + (2n+2m-8) \cdot 5 + (mn-2n-2m+4) \cdot 8 = 8mn - 6n - 6m + 4.$$

This number is exactly twice the number of handshakes, because each handshake involves two students. Hence, we see that $8mn - 6n - 6m + 4 = 2040$, or $16mn - 12n - 12m = 4072$. We can rewrite this equation as

$$(4m-3) \cdot (4n-3) = 4081.$$

The integers $4m-3$ and $4n-3$ both have remainder 1 when dividing by 4. Also, $4m-3$ and $4n-3$ must both be greater than 1. If we look at the prime factorisation $4081 = 7 \cdot 11 \cdot 53$, then we see that there is only one way to write 4081 as the product of two integers greater than 1 that each have remainder 1 when dividing by 4: namely as $4081 = 77 \cdot 53$. Hence, we find $4m-3 = 77$ and $4n-3 = 53$ (or the other way around). It follows that $m = 20$ and $n = 14$ (or the other way around). In both cases we see that the total number of students equals $20 \cdot 14 = 280$.

Final Round, September 2016

Problems

1. (a) On a long pavement, a sequence of 999 integers is written in chalk. The numbers need not be in increasing order and need not be distinct. Merlijn encircles 500 of the numbers with red chalk. From left to right, the numbers circled in red are precisely the numbers $1, 2, 3, \ldots, 499, 500$. Next, Jeroen encircles 500 of the numbers with blue chalk. From left to right, the numbers circled in blue are precisely the numbers $500, 499, 498, \ldots, 2, 1$.
Prove that the middle number in the sequence of 999 numbers is circled both in red and in blue.

(b) Merlijn and Jeroen cross the street and find another sequence of 999 integers on the pavement. Again Merlijn circles 500 of the numbers with red chalk. Again the numbers circled in red are precisely the numbers $1, 2, 3, \ldots, 499, 500$ from left to right. Now Jeroen circles 500 of the numbers, not necessarily the same as Merlijn, with green chalk. The numbers circled in green are also precisely the numbers $1, 2, 3, \ldots, 499, 500$ from left to right.
Prove: there is a number that is circled both in red and in green that is *not* the middle number of the sequence of 999 numbers.

2. For an integer $n \geq 1$ we consider sequences of $2n$ numbers, each equal to 0, -1 or 1. The *sum product value* of such a sequence is calculated by first multiplying each pair of numbers from the sequence, and then adding all the results together.

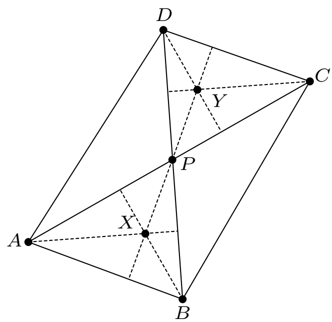
For example, if we take $n = 2$ and the sequence $0, 1, 1, -1$, then we find the products $0 \cdot 1$, $0 \cdot 1$, $0 \cdot -1$, $1 \cdot 1$, $1 \cdot -1$, $1 \cdot -1$. Adding these six results gives the sum product value of this sequence: $0+0+0+1+(-1)+(-1) = -1$. The sum product value of this sequence is therefore smaller than the sum product value of the sequence $0, 0, 0, 0$, which equals 0.
Determine for each integer $n \geq 1$ the smallest sum product value that such a sequence of $2n$ numbers could have.
Attention: you are required to prove *that a smaller sum product value is impossible.*

3. Find all possible triples (a, b, c) of positive integers with the following properties:

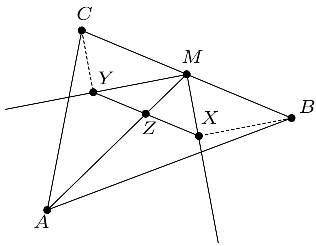
- $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$;
- a is a divisor of $a + b + c$;
- b is a divisor of $a + b + c$;
- c is a divisor of $a + b + c$.

(Here $\gcd(x, y)$ is the greatest common divisor of x and y.)

4. Version for junior students
In a quadrilateral $ABCD$ the intersection of the diagonals is called P. Point X is the orthocentre of triangle PAB. (The orthocentre of a triangle is the point where the three altitudes of the triangle intersect.) Point Y is the orthocentre of triangle PCD. Suppose that X lies inside triangle PAB and Y lies inside triangle PCD. Moreover, suppose that P is the midpoint of line segment XY. Prove that $ABCD$ is a parallelogram.



4. Version for senior students
In the acute triangle ABC, the midpoint of side BC is called M. Point X lies on the angle bisector of $\angle AMB$ such that $\angle BXM = 90°$. Point Y lies on the angle bisector of $\angle AMC$ such that $\angle CYM = 90°$. Line segments AM and XY intersect in point Z.
Prove that Z is the midpoint of XY.



5. Bas has coloured each of the positive integers. He had several colours at his disposal. His colouring satisfies the following requirements:

- each odd integer is coloured blue;
- each integer n has the same colour as $4n$;
- each integer n has the same colour as at least one of the integers $n + 2$ and $n + 4$.

Prove that Bas has coloured all integers blue.

Solutions

1. (a) For brevity, we will say that a number encircled in red is a red number, and similarly for blue. So some numbers could be both red and blue. Since there are 999 numbers written on the pavement, of which 500 are red and 500 are blue, we have at least one bicoloured number by the pigeonhole principle. Consider such a bicoloured number and suppose it is the number k. From left to right, the red numbers form the sequence $1, 2, \ldots, 500$. Hence, to the left of the bicoloured number we have the red numbers 1 to $k-1$, and to the right we have the red numbers $k+1$ to 500. The blue numbers are written in the opposite order: from left to right they form the sequence $500, 499, \ldots, 1$. To the left of the bicoloured number we therefore have the blue numbers $k+1$ to 500, and to the right we have the blue numbers 1 to $k-1$.

We count how many numbers there are on each side of the bicoloured number. On the left we have the red numbers 1 to $k-1$ and the blue numbers $k+1$ to 500. Hence, there are at least 499 distinct numbers on the left. On the right we have the red numbers $k+1$ to 500 and the blue numbers 1 to $k-1$. Again at least 499 distinct numbers. Since there are only $999 = 499 + 1 + 499$ numbers on the pavement, we have already considered all numbers on the pavement. We conclude that there are precisely 499 numbers on each side of the bicoloured number. The bicoloured number is therefore precisely in the middle of the sequence.

(b) Just as in part (a), the pigeonhole principle yields that at least one number is bicoloured, i.e. both red and green. If more than one number is bicoloured, one of them is not exactly in the middle of the sequence and we are done. Therefore, it suffices to examine the case where there is exactly one bicoloured number. Let this number be k.

Again, we count how many numbers there are on each side of the bicoloured number. On the left we have the red numbers 1 to $k-1$ and the green numbers 1 to $k-1$. Since none of these numbers is bicoloured, there are at least $2 \cdot (k-1)$ distinct numbers on the left. On the right we have the red numbers $k+1$ to 500 and the green numbers $k+1$ to 500. Since none of these numbers is bicoloured, we have at least $2 \cdot (500-k)$ distinct numbers on the right. Since $2 \cdot (k-1) + 1 + 2 \cdot (500-k) = 999$, we have already counted all numbers.

On the left of the bicoloured number we therefore have exactly $2 \cdot (k-1)$ numbers, and on the right we have exactly $2 \cdot (500-k)$ numbers. Since $2 \cdot (k-1)$ is an *even* number, it is unequal to 499. We conclude that the bicoloured number is not exactly in the middle of the sequence.

2. Suppose that our sequence has x ones, y minus ones (and hence $2n - x - y$ zeroes). We calculate the sum product value of the sequence (as an expression in x and y).

In the sum product value, six different types of terms occur: $1 \cdot 1$, $1 \cdot -1$, $-1 \cdot -1$, $1 \cdot 0$, $-1 \cdot 0$, and $0 \cdot 0$. Only the first three types contribute since the other types are equal to 0.

The number of terms of the type $1 \cdot 1 = 1$ equals the number of ways to select two out of x ones. This can be done in $\frac{x(x-1)}{2}$ ways: there are x options for the first 1, and then $x - 1$ options for the second 1. Since the order in which we select the two ones does not matter, we effectively count each possible pair twice.

Similarly, the number of terms of the type $-1 \cdot -1 = 1$ is equal to $\frac{y(y-1)}{2}$.

The number of terms of the type $1 \cdot -1 = -1$ is equal to xy, since there are x options for choosing a 1 and, independently, there are y options for choosing a -1.

In total, we obtain a sum product value of

$$S = \frac{x(x-1)}{2} \cdot 1 + \frac{y(y-1)}{2} \cdot 1 + xy \cdot -1 = \frac{(x-y)^2 - (x+y)}{2}.$$

Since $(x-y)^2 \geq 0$ (squares are non-negative) and $-(x+y) \geq -2n$ (there are only $2n$ numbers in the sequence), we see that $S \geq \frac{0-2n}{2} = -n$. So the sum product value cannot be smaller than $-n$. If we now choose $x = y = n$, then $(x-y)^2 = 0$ and $-x - y = -2n$, which imply a sum product value of exactly $\frac{0-2n}{2} = -n$. Hence, the smallest possible sum product value is $-n$.

3. The problem is symmetric in a, b, and c. That is, if we consistently swap a and b, or a and c, or b and c, then the conditions on (a, b, c) do not change. We will therefore consider solutions for which $a \leq b \leq c$. The remaining solutions are then found by permuting the values of a, b, and c.

Since a and b are positive, we see that $a+b+c > c$. Since c is largest among the three numbers, we also have $a + b + c \leq 3c$. Since we are given that $a+b+c$ is a multiple of c, we are left with two possibilities: $a+b+c = 2c$ or $a + b + c = 3c$. We consider both cases separately. If $a + b + c = 3c$, then a, b, and c must all be equal, because otherwise, the fact that $a, b \leq c$ implies that $a+b+c < 3c$. This means that $\gcd(b, c) = \gcd(c, c) = c$. Since $\gcd(b, c)$ must be equal to 1, we find $(a, b, c) = (1, 1, 1)$. This is indeed a solution, since $\gcd(1, 1) = 1$ and 1 is a divisor of $1 + 1 + 1$.

If $a+b+c=2c$, then $c=a+b$. We know that b must be a divisor of $a+b+c=2a+2b$. Since $a>0$, we have $2a+2b>2b$. Since $b \geq a$, we also have $2a+2b \leq 4b$. Therefore, since $2a+2b$ must be a multiple of b, there are only two possibilities: $2a+2b=3b$ or $2a+2b=4b$. Again, we consider these cases separately.

If $2a+2b=3b$, then $b=2a$. Similarly to the first case, we find that $a=\gcd(a,2a)=\gcd(a,b)=1$. Therefore, $b=2$ and $c=a+b=3$. The resulting triple $(a,b,c)=(1,2,3)$ is indeed a solution, since $\gcd(1,2)=\gcd(1,3)=\gcd(2,3)=1$ and $1+2+3=6$ is divisible by 1, 2, and 3.

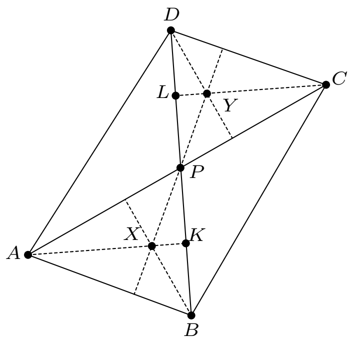
If $2a+2b=4b$, then $a=b$. Again we see that $a=\gcd(a,a)=\gcd(a,b)=1$. From $b=a=1$ it follows that $c=a+b=2$ and hence $(a,b,c)=(1,1,2)$. This is indeed a solution since $\gcd(1,1)=\gcd(1,2)=1$ and $1+1+2=4$ is divisible by 1 and 2.

We conclude that the solutions for which $a \leq b \leq c$ holds are: $(a,b,c)=(1,1,1)$, $(a,b,c)=(1,1,2)$, and $(a,b,c)=(1,2,3)$. Permuting the values of a, b and c, we obtain a total of ten solutions (a,b,c):

$$(1,1,1), \quad (1,1,2),\ (1,2,1),\ (2,1,1),$$

$$(1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1).$$

4. Version for junior students



Let K be the intersection of AX and BD, and let L be the intersection of CY and BD. Consider the triangles PLY and PKX. The angles $\angle PLY$ and $\angle PKX$ are both right angles. The angles $\angle YPL$ and $\angle XPK$ are opposite angles and therefore equal. Since $|PX|=|PY|$, we see that the triangles PKX and PLY are congruent (SAA). Hence, we have $|PK|=|PL|$.

Now consider triangles PAK and PCL. Angles $\angle AKP$ and $\angle CLP$ are both right angles. Angles $\angle KPA$ and $\angle LPC$ are opposite angles, hence equal. We have already shown that $|PK|=|PL|$. Therefore, triangles PAK and PCL are congruent (ASA). From this, we conclude that $|AP|=|PC|$.

In a similar fashion, we may deduce that $|BP|=|DP|$. The two diagonals of $ABCD$ bisect each other, hence $ABCD$ is a parallelogram.

4. Version for senior students

We start by observing that we have that $\angle CMY = \frac{1}{2}\angle CMA = \frac{1}{2}(180° - \angle AMB) = 90° - \angle XMB$. Since $\angle MXB = 90°$, and the angles of triangle BMX sum to $180°$, we see that $\angle CMY = 90° - \angle XMB = \angle MBX$.

Looking at triangles CMY and MBX, we observe that $\angle CMY = \angle MBX$, $\angle MYC = 90° = \angle BXM$, and $|CM| = |MB|$. The two triangles are therefore congruent (SAA). In particular, we obtain the equalities $|MX| = |CY|$ and $|MY| = |BX|$.

Now consider triangle XYM. We already know that $|MY| = |BX|$ and that $\angle YMX = \angle YMA + \angle AMX = \frac{1}{2}\angle CMA + \frac{1}{2}\angle AMB = \frac{1}{2} \cdot 180° = 90°$. Since triangles XYM and MBX also share the side MX, they are congruent (SAS).

In particular, we see that $\angle MXY = \angle XMB$. Since MX is the angle bisector of $\angle AMB$, we have $\angle XMB = \angle AMX$. This implies that triangle MXZ has two equal angles and is therefore an isosceles triangle with vertex angle Z. We conclude that $|MZ| = |XZ|$.

In a similar manner, we see that triangles XYM and CMY are congruent, and find that $\angle XYM = \angle CMY = \angle YMA$. Triangle MYZ is therefore isosceles with vertex angle Z. This implies that $|YZ| = |MZ|$. Together with $|MZ| = |XZ|$, this concludes the proof.

5. Suppose that not all numbers are coloured blue. Then, there must be a number k that is not blue. We will use this to derive a contradiction.

Without loss of generality, we may assume that k is coloured red. Since all *odd* numbers are blue, k must be *even*, say $k = 2m$ for some integer $m \geq 1$. From the second requirement, it follows that $8m$ is red as well. From the third requirement, it now follows that at least one of the numbers $8m + 2$ and $8m + 4$ is red. However, $2m + 1$ is odd and therefore blue, which by the second requirement implies that $8m + 4 = 4 \cdot (2m + 1)$ is blue as well. So $8m + 2$ must be red.

By the third requirement, $8m - 2$ must be the same colour as $8m$ or $8m + 2$. Since both $8m$ and $8m + 2$ are red, this implies that $8m - 2$ must be red as well. Since $8m$ and $8m - 2$ are red, this implies (again by the third requirement) that $8m - 4$ is also red. The second requirement now implies that $(8m - 4)/4 = 2m - 1$ is also red. But that is impossible since $2m - 1$ is odd, and therefore blue.

We conclude that the assumption that not all numbers are blue leads to a contradiction. Therefore, all numbers must be blue.

BxMO Team Selection Test, March 2017

Problems

1. Let n be an even positive integer. A sequence of n real numbers is called *complete* if for every integer m with $1 \leq m \leq n$ either the sum of the first m terms of the sum or the sum of the last m terms is integral. Determine the minimum number of integers in a complete sequence of n numbers.

2. A function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}$ is given, which has the following properties:

(i) $f(p) = 1$ for all prime numbers p,

(ii) $f(xy) = yf(x) + xf(y)$ for all $x, y \in \mathbb{Z}_{>0}$.

Determine the smallest $n \geq 2016$ satisfying $f(n) = n$.

3. Let ABC be a triangle with $\angle A = 90^\circ$ and let D be the orthogonal projection of A onto BC. The midpoints of AD and AC are called E and F, respectively. Let M be the circumcentre of $\triangle BEF$. Prove that $AC \parallel BM$.

4. A quadruple (a, b, c, d) of positive integers with $a \leq b \leq c \leq d$ is called *good* if we can colour each integer red, blue, green or purple, in such a way that

- of each a consecutive integers at least one is coloured red;
- of each b consecutive integers at least one is coloured blue;
- of each c consecutive integers at least one is coloured green;
- of each d consecutive integers at least one is coloured purple.

Determine all good quadruples with $a = 2$.

5. Determine all pairs of prime numbers (p, q) such that $p^2 + 5pq + 4q^2$ is the square of an integer.

Solutions

1. We will prove that the minimum number of integers in a complete sequence is 2. First consider the case $n = 2$. Let a_1 and a_2 be the numbers in the sequence. Then either a_1 or a_2 is integral. Without loss of generality assume a_1 is integral. Moreover, $a_1 + a_2$ is integral, hence also a_2 is integral. Therefore, the sequence contains at least two integers.

Now consider the case $n > 2$. Write $n = 2k$ (because n is even) with $k \geq 2$. Then either $a_1 + a_2 + \ldots + a_k$ or $a_{k+1} + a_{k+2} + \ldots + a_{2k}$ is integral. But, as the sum of both expressions is also integral, they are both integral. Moreover, either $a_1 + a_2 + \ldots + a_{k-1}$ or $a_{k+2} + a_{k+3} + \ldots + a_{2k}$ is integral. This yields that either a_k or a_{k+1} is integral. Moreover, we know that a_1 or a_{2k} is integral, and these do not coincide with a_k or a_{k+1} because $k \geq 2$. Hence, at least two different numbers are integers.

Finally, we will show for each even integer n that it is possible to write a complete sequence with exactly two integers. Again write $n = 2k$. If k is odd, we take $a_1 = a_{k+1} = 1$ and all other terms equal to $\frac{1}{2}$. The sum of all numbers in the sequence is integral, hence it is sufficient to show that the sum of the first or last m terms is integral for $1 \leq m \leq k$; the cases in which $m > k$ follow directly. For odd $m \leq k$, the sum of the first m terms is integral, for even $m < k$, the sum of the last m terms is integral.

If k is even, we take $a_1 = a_k = 1$ and all other terms equal to $\frac{1}{2}$. For odd $m < k$, the sum of the first m terms is integral, and for even $m \leq k$, the sum of the last m terms is integral. Moreover, the sum of all terms is integral, hence the requirement is also met for $m > k$.

We conclude that the minimum number of integers in a complete sequence of n numbers is 2. $\square$

2. We will first prove that for prime numbers p and positive integers k we have $f(p^k) = kp^{k-1}$. We will prove this using mathematical induction to k. For $k = 1$, the statement becomes $f(p) = 1$, which is known to hold. Now, let $l \geq 1$ and suppose that we proved the statement for $k = l$. Consider $k = l + 1$. Then we apply the second property with $x = p$ and $y = p^l$:

$$f\left(p^{l+1}\right) = f\left(p \cdot p^l\right) = p^l \cdot f(p) + p \cdot f\left(p^l\right) = p^l + p \cdot lp^{l-1} = (l+1)p^l.$$

This finished the induction. Now we will prove that for distinct prime numbers $p_1, p_2, \ldots, p_t$ and positive integers $a_1, a_2, \ldots, a_t$ we have

$$f\left(p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}\right) = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \cdot \left(\frac{a_1}{p_1} + \frac{a_2}{p_2} + \cdots + \frac{a_t}{p_t}\right).$$

We will prove this by induction to t. For $t = 1$, this is exactly the formula that we have just proven. Now, let $r \geq 1$ and suppose that the statement is proved for $t = r$. Then we apply the second property again:

$$\begin{aligned}
&f\left(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \cdot p_{r+1}^{a_{r+1}}\right) \\
&\quad = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \cdot f\left(p_{r+1}^{a_{r+1}}\right) + p_{r+1}^{a_{r+1}} \cdot f\left(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}\right) \\
&\quad = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \cdot a_{r+1} p_{r+1}^{a_{r+1}-1} + p_{r+1}^{a_{r+1}} \cdot p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \cdot \left(\tfrac{a_1}{p_1} + \tfrac{a_2}{p_2} + \cdots + \tfrac{a_r}{p_r}\right) \\
&\quad = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}} \cdot \tfrac{a_{r+1}}{p_{r+1}} + p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}} \cdot \left(\tfrac{a_1}{p_1} + \tfrac{a_2}{p_2} + \cdots + \tfrac{a_r}{p_r}\right) \\
&\quad = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}} \cdot \left(\tfrac{a_1}{p_1} + \tfrac{a_2}{p_2} + \cdots + \tfrac{a_r}{p_r} + \tfrac{a_{r+1}}{p_{r+1}}\right).
\end{aligned}$$

This finishes the proof by induction.

For an integer $n > 1$ of the shape $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ with p_i distinct and prime, the equality $f(n) = n$ is equivalent to

$$p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \cdot \left(\tfrac{a_1}{p_1} + \tfrac{a_2}{p_2} + \cdots + \tfrac{a_t}{p_t}\right) = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t},$$

or

$$\tfrac{a_1}{p_1} + \tfrac{a_2}{p_2} + \cdots + \tfrac{a_t}{p_t} = 1.$$

Multiplying this with $p_1 p_2 \cdots p_t$ yields

$$a_1 p_2 p_3 \cdots p_t + a_2 p_1 p_3 \cdots p_t + \ldots + a_t p_1 p_2 \cdots p_{t-1} = p_1 p_2 \cdots p_t.$$

Assume without loss of generality that p_1 is the smallest prime divisor of n. In the expression above, p_1 is a divisor of the right hand side and of each term in the left hand side except for the first term possibly. But then p_1 must also be a divisor of the first term. As $p_2, \ldots, p_t$ are all prime numbers unequal to p_1, this is only possible when $p_1 \mid a_1$. In particular, we have $a_1 \geq p_1$, yielding $\frac{a_1}{p_1} \geq 1$. We now see that equality must hold here, and hence $t = 1$, because the sum of the $\frac{a_i}{p_i}$ would otherwise be greater than 1. Hence, $n = p^p$ for a certain prime number p.

Now we are searching for the smallest $n \geq 2016$ of the shape $n = p^p$. As $3^3 = 27$ and $5^5 = 3125$, this smallest n is 3125. □

3. Because of the right angles at A and D we have $\triangle ADB \sim \triangle CAB$ (AA). This yields that $\frac{|AD|}{|AB|} = \frac{|CA|}{|CB|}$. Because $|AE| = \frac{1}{2}|AD|$ and $|CF| = \frac{1}{2}|CA|$, we also have that $\frac{|AE|}{|AB|} = \frac{|CF|}{|CB|}$. From the previous similarity, we also obtain $\angle BAE = \angle BAD = \angle BCA = \angle BCF$. Hence, using the (SAS) criterion, we get $\triangle AEB \sim \triangle CFB$, from which we obtain $\angle ABE = \angle CBF$.

Moreover, EF is a mid-parallel in triangle ADC, hence $EF \parallel BC$. Therefore, we have $\angle CBF = \angle BFE$. Using the inscribed angle theorem and the sum of the angles in the isosceles triangle EBM, we obtain $\angle BFE = \frac{1}{2}\angle BME = 90^\circ - \angle EBM$. Therefore, $\angle ABE = \angle CBF = 90^\circ - \angle EBM$. We conclude that $\angle ABM = \angle ABE + \angle EBM = 90^\circ$, and $AC \parallel BM$. $\square$

4. Each time, we only consider quadruples (a, b, c, d) of positive integers satisfying $a \leq b \leq c \leq d$. Each quadruple with $b \geq 6$ meets the requirements: in this case, we can colour the even numbers red, and the odd numbers consecutively blue, green, purple, blue, green, purple, et cetera. Moreover, each quadruple with $b \geq 4$ and $c \geq 8$ meets the requirements: in this case, we can colour the even numbers red, the numbers 1 mod 4 blue, the numbers 3 mod 8 green, and the numbers 7 mod 8 purple. We will prove that these are the only possibilities.

Suppose that $(2, b, c, d)$ is good. We consider a purple number. Its neighbours must be red, so then we have 3 consecutive numbers of which none is coloured blue. Hence, $b \geq 4$. If $b \geq 6$, then the quadruple is good, as we've just seen. Now suppose that either $b = 4$ or $b = 5$. We were considering a purple number with its two red neighbours. On at least one of both ends there must be a blue number, say on the left side. Right left of this blue number, there is another red number. So we got $RBRPR$. Of the next two numbers in the sequence, there is at least one red one and at least one blue one, so there is certainly no green one. In this way, we found 7 consecutive numbers none of which is coloured green, hence $c \geq 8$. We have already seen that the quadruple is good in this case.

We conclude that the following quadruples $(2, b, c, d)$ are good quadruples: these with either $b \geq 6$, or with $b \geq 4$ and $c \geq 8$. $\square$

5. Write $p^2 + 5pq + 4q^2 = a^2$, for an integer $a \geq 0$. The left hand side equals $(p + 2q)^2 + pq$, hence we can rewrite this to $pq = a^2 - (p + 2q)^2$, or $pq = (a - p - 2q)(a + p + 2q)$. The second factor on the right hand side is greater than p and greater than q, but it is a divisor of pq. Because p and q are prime, it must be equal to pq, hence $a + p + 2q = pq$. Then, we have $a - p - 2q = 1$. Subtracting these two equalities, we obtain $pq - 1 = (a + p + 2q) - (a - p - 2q) = 2(p + 2q)$, or $pq - 2p - 4q - 1 = 0$. This is equivalent to $(p - 4)(q - 2) = 9$. The factor $q - 2$ cannot be negative, hence also $p - 4$ is not negative. The factors must be equal to 1 and 9, or to 9 and 1, or to 3 and 3. For (p, q), this yields the possibilities $(5, 11)$, $(13, 3)$ and $(7, 5)$, respectively. We can check that $p^2 + 5pq + 4q^2$ is a square in each of these cases, namely 28^2, 20^2 and 18^2, respectively. Therefore, these three pairs are the solutions. $\square$

IMO Team Selection Test 1, June 2017

Problems

1. Let n be a positive integer. Suppose that we have disks of radii $1, 2, \ldots, n$. Of each size there are two disks: a transparent one and an opaque one. In every disk there is a small hole in the centre, with which we can stack the disks using a vertical stick. We want to make stacks of disks that satisfy the following conditions:

- Of each size exactly one disk lies in the stack.
- If we look at the stack from directly above, we can see the edges of all of the n disks in the stack. (So if there is an opaque disk in the stack, no smaller disks may lie beneath it.)

Determine the number of distinct stacks of disks satisfying these conditions. (Two stacks are distinct if they do not use the same set of disks, or, if they do use the same set of disks and the orders in which the disks occur are different.)

2. Let $n \geq 4$ be an integer. Consider a regular $2n$-gon for which to every vertex, an integer is assigned, which we call the *value* of said vertex. If four distinct vertices of this $2n$-gon form a rectangle, we say that the sum of the values of these vertices is a *rectangular sum*.

Determine for which (not necessarily positive) integers m the integers $m+1, m+2, \ldots, m+2n$ can be assigned to the vertices (in some order) in such a way that every rectangular sum is a prime number. *(Prime numbers are positive by definition.)*

3. Determine all possible values of $\frac{1}{x} + \frac{1}{y}$ if x and y are non-zero real numbers satisfying $x^3 + y^3 + 3x^2y^2 = x^3y^3$.

4. Let ABC be a triangle, let M be the midpoint of AB, and let N be the midpoint of CM. Let X be a point satisfying both $\angle XMC = \angle MBC$ and $\angle XCM = \angle MCB$ such that X and B lie on opposite sides of CM. Let Ω be the circumcircle of triangle AMX.

(a) Show that CM is tangent to Ω.

(b) Show that the lines NX and AC intersect on Ω.

Solutions

1. We say that a stack of disks is *valid* if it satisfies the conditions. Let a_n denote the number of valid stacks with n disks (of radii $1, 2, \ldots, n$). We show by induction on n that $a_n = (n+1)!$. For $n = 1$, we note that we can make two distinct stacks, namely one with the transparent disk, and one with the opaque one. So suppose that for some $n \geq 1$, we have shown that $a_n = (n+1)!$. Consider a valid stack with $n+1$ disks. If we remove the disk with radius $n+1$, the edge of every disk is still visible from directly above, so we are left with a valid stack of n disks. Therefore, every valid stack of $n+1$ disks can be made by adding a disk of radius $n+1$ to a valid stack of n disks at some location. This can in principle be done in $n+1$ ways: above the top disk, above the second highest disk, ..., above the bottom disk, and under the bottom disk. The disk of radius $n+1$ is always visible, regardless of where in the stack it is inserted. If it is inserted under the bottom disk, it may both be transparent or opaque, so there are $2a_n$ valid stacks of $n+1$ disks in which the disk of radius $n+1$ is the bottom disk. If an opaque disk of radius $n+1$ is inserted anywhere but under the bottom disk, the disks below this opaque disk will now longer be visible. So in the remaining n locations, we can only insert the transparant disk of radius $n+1$. Therefore there are na_n valid stacks in which the disk of radius $n+1$ is not the bottom disk. Hence

$$a_{n+1} = 2a_n + na_n = (n+2)a_n = (n+2)(n+1)! = (n+2)!,$$

which completes the induction. □

2. Number the vertices of the $2n$-gon from 1 up to $2n$, clockwise, and let a_i be the value of vertex i. Since the number of vertices of the polygon is even, we can pair each vertex to the one directly opposite to it. Sum the values of each pair of opposite vertices to get the numbers $s_i = a_i + a_{i+n}$.

If four vertices A, B, C, and D lie in that order on a rectangle, then $\angle ABC = 90°$ and $\angle ADC = 90°$, so B and D lie on the circle with diameter AC by Thales's theorem. As all vertices of the polygon lie on its circumcircle, AC is the diameter of this circumcircle. Therefore A and C are opposite vertices, and the same holds for B and D. Conversely, if A and C are opposite vertices, and so are B and D, then $ABCD$ is a rectangle, again by Thales's theorem. In short, four vertices form a rectangle if and only if they form two pairs of opposite vertices.

So the condition in the problem is equivalent to $s_i + s_j$ being prime for all $1 \leq i < j \leq n$. Suppose that at least three of the s_i have the same parity, say s_j, s_k, and s_l. Then the sum of each pair of these three numbers is even, but prime as well, so this sum must be 2 in all cases. Hence $s_j + s_k = s_k + s_l = s_j + s_l$, from which follows that $s_j = s_k = s_l$. Since $s_j + s_k = 2$, we have $s_j = s_k = s_k = 1$. Therefore at most two of the s_i are even, and if three or more of the s_i are odd, then the odd s_i all equal 1.

The sum of all s_i is equal to the sum of the values of the vertices, which is

$$(m+1)+(m+2)+\cdots+(m+2n) = 2mn+\tfrac{1}{2}\cdot 2n\cdot(2n+1) = 2mn+n(2n+1).$$

On the one hand, this sum is congruent to n modulo 2. On the other hand, modulo 2 this sum must be congruent tot the number of odd s_i. Hence the number of odd s_i has the same parity as n. As there are n of the s_i, it follows that the number of even s_i is even. Since we have already seen that the number of even s_i is at most 2, it follows this number is either 0 or 2.

Suppose that $n = 4$ and and that there are two even s_i. Then there are also two odd s_i. The sum of the even s_i, as well as that of the odd s_i, is equal to 2, so the sum of the s_i is equal to $4 = n$. Now suppose that $n \geq 5$ and that there are two even s_i. Then $n - 2 \geq 3$ of the s_i are odd, so all odd s_i are equal to 1. The sum of the two even s_i is 2, so the sum of the s_i is equal to $(n-2)+2 = n$. Now suppose that there are no even s_i. Then all s_i are odd, and therefore equal to 1, so again, the sum of the s_i equals n.

So in all of the cases, the sum of the s_i is n. We have also seen that this sum is equal to $2mn + n(2n+1)$. Therefore $2mn + n(2n+1) = n$, so dividing by the non-zero integer n gives the equality $2m + 2n + 1 = 1$. Hence $m = -n$.

Finally, we show that if $m = -n$, there indeed exists a solution. Let $a_i = i$ and $a_{n+i} = 1 - i$ for $1 \leq i \leq n$. Then the integers $1, 2, \ldots, n$ and the integers $0, -1, \ldots, -n+1$ are values of vertices; these are precisely the integers $m+1, m+2, \ldots, m+2n$ for $m = -n$. Moreover, we have $s_i = i + (1-i) = 1$ for all i. Hence all rectangular sums are equal to 2, which is a prime number.

Therefore $m = -n$ is the only value of m for which the there exists a solution. $\square$

3. We rewrite the equation as $x^3 + y^3 - x^3y^3 = -3x^2y^2$, which gives

$$\begin{aligned}(x+y)^3 - x^3y^3 &= x^3 + 3x^2y + 3xy^2 + y^3 - x^3y^3\\ &= -3x^2y^2 + 3x^2y + 3xy^2 = 3xy(-xy + x + y).\end{aligned}$$

Using the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ we can also write this as

$$\begin{aligned}(x+y)^3 - x^3y^3 &= (x+y-xy)\big((x+y)^2 + xy(x+y) + (xy)^2\big)\\ &= (x+y-xy)(x^2+y^2+2xy+x^2y+xy^2+x^2y^2).\end{aligned}$$

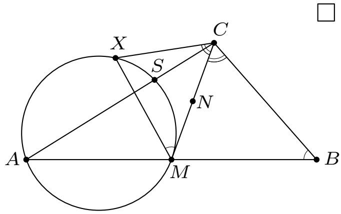
Therefore the two right hand sides are equal, so we see that either $x+y-xy = 0$ or $x^2+y^2+2xy+x^2y+xy^2+x^2y^2 = 3xy$. The latter case can be rewritten as

$$\begin{aligned}0 &= x^2+y^2-xy+x^2y+xy^2+x^2y^2\\ &= \tfrac{1}{2}(x-y)^2 + \tfrac{1}{2}x^2(y^2+2y+1) + \tfrac{1}{2}y^2(x^2+2x+1)\\ &= \tfrac{1}{2}(x-y)^2 + \tfrac{1}{2}x^2(y+1)^2 + \tfrac{1}{2}y^2(x+1)^2.\end{aligned}$$

So since the sum of three squares must be 0, each of the squares must be 0 as well. Hence $x = y = -1$. This indeed gives a solution of the given equation, and we have $\frac{1}{x} + \frac{1}{y} = 2$.

In the remaining case we have $x+y-xy = 0$, so $\frac{1}{y} + \frac{1}{x} - 1 = 0$, from which we deduce that $\frac{1}{x} + \frac{1}{y} = 1$. This is attained for example for $x = y = 2$. Therefore the possible values are -2 and 1. $\square$

4. We consider the configuration in the figure; the proof is similar for the other configurations.



(a) First, we have $\triangle XMC \sim \triangle MBC$.
We have $\angle AMX = 180^\circ - \angle XMC - \angle BMC = 180^\circ - \angle XMC - \angle MXC = \angle MCX$, and $\frac{|AM|}{|MX|} = \frac{|BM|}{|MX|} = \frac{|MC|}{|CX|}$, so we have $\triangle AMX \sim \triangle MCX$. Hence $\angle XAM = \angle XMC$, so using the converse of the tangent chord angle theorem we see that CM is tangent to Ω.

(b) Let S be the second intersection point of AC and Ω (or the point at which AC and Ω are tangent; which then would be A). We need to show that S lies on NX. By the tangent chord angle theorem we have $\angle SMC = \angle SAM = \angle CAM$. (If $S = A$, then $\angle SMC = \angle CAM$ follows from CA and CM both being tangents, as then $\triangle CAM$ is isosceles.) Therefore $\triangle CSM \sim \triangle CMA$ (AA), so $\frac{|CM|}{|CA|} = \frac{|SM|}{|MA|}$. Since $|CM| = 2|MN|$ and $|MA| = \frac{1}{2}|AB|$, it follows that $\frac{|MN|}{|CA|} = \frac{|SM|}{|AB|}$. Togther with $\angle SMN = \angle SMC = \angle CAM = \angle CAB$, we get $\triangle SNM \sim \triangle BCA$. Hence $\angle MSN = \angle ABC$. Moreover, we have $\angle XSM = 180^\circ - \angle XAM$. In (a), we have seen that $\angle XAM = \angle XMC = \angle MBC = \angle ABC$, so $\angle XSM = 180^\circ - \angle ABC = 180^\circ - \angle MSN$. Therefore S lies on XN. $\square$

IMO Team Selection Test 2, June 2017

Problems

1. Let a, b, and c be distinct positive integers, and suppose that $p = ab+bc+ca$ is a prime number.

(a) Show that a^2, b^2, and c^2 give distinct remainders after division by p.

(b) Show that a^3, b^3, and c^3 give distinct remainders after division by p.

2. The incircle of a non-isosceles triangle $\triangle ABC$ has centre I and is tangent to BC and CA in D and E, respectively. Let H be the orthocentre of $\triangle ABI$, let K be the intersection of AI and BH and let L be the intersection of BI and AH. Show that the circumcircles of $\triangle DKH$ and $\triangle ELH$ intersect on the incircle of $\triangle ABC$.

3. Let $k > 2$ be an integer. A positive integer ℓ is said to be *k-pable* if the numbers $1, 3, 5, \ldots, 2k-1$ can be partitioned into two subsets A and B in such a way that the sum of the elements of A is exactly ℓ times as large as the sum of the elements of B.

Show that the smallest k-pable integer is coprime to k.

4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(y+1)f(x) + f\big(xf(y) + f(x+y)\big) = y$$

for all $x, y \in \mathbb{R}$.

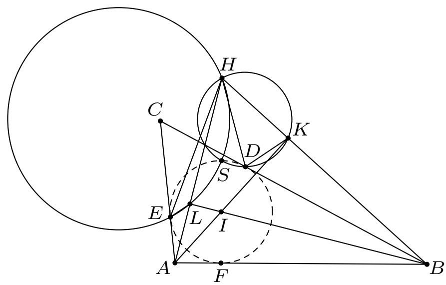
Solutions

1. (a) Suppose for a contradiction that two of a^2, b^2, and c^2 have the same remainder after division by p, say $a^2 \equiv b^2 \mod p$. Then $p \mid a^2 - b^2 = (a-b)(a+b)$, so $p \mid a - b$ or $p \mid a + b$. In the latter case, we have $p \leq a + b \leq c(a+b) < ab + bc + ca = p$, which is a contradiction. In the former case, we see that by $a \neq b$, we have $p \leq |a - b| \leq a + b < p$, which is a contradiction.

(b) Suppose for a contradiction that two of a^3, b^3, and c^3 have the same remainder after division by p, say $a^3 \equiv b^3 \mod p$. Then $p \mid a^3 - b^3 = (a-b)(a^2+ab+b^2)$, so $p \mid a-b$ or $p \mid a^2+ab+b^2$. The former case leads to a contradiction, as we have seen in (a). Consider the remaining case: $p \mid a^2+ab+b^2$. Then $p \mid a^2+ab+b^2+(ab+bc+ca) = (a+b)(a+b+c)$, so $p \mid a+b$ or $p \mid a+b+c$. However, note that $a+b < a+b+c < ab+bc+ca$, since a, b, and c cannot all be equal to 1. Therefore p is neither a divisor of $a+b$, nor $a+b+c$, which is a contradiction.

$\square$

2. Consider the configuration in the figure; the solution is similar for the other configurations.



We have $\angle IDB = 90° = \angle IKB$, so $BKDI$ is a cyclic quadrilateral. Moreover, we have $\angle ALB = 90° = \angle AKB$, so $BKLA$ is also a cyclic quadrilateral. So

$$\angle BKD = 180° - \angle BID = 180° - (90° - \tfrac{1}{2}\angle ABC)$$
$$= 180° - \angle BAL = \angle BKL.$$

Hence K, D, and L are collinear. Analogously, K, E, and L are collinear, from which we deduce that all four of K, D, E, and L are collinear.

Let S be the second intersection point of the circumcircles of $\triangle DKH$ and $\triangle ELH$. Then we have

$$\angle DSE = 360° - \angle DSH - \angle HSE = \angle DKH + 180° - \angle HLE$$
$$= \angle LKH + \angle HLK = 180° - \angle KHL.$$

Since $HLIK$ is also a cyclic quadrilateral (as it has two opposite right angles) we have

$$\begin{aligned}180^\circ - \angle KHL = \angle KIL = \angle AIB &= 180^\circ - \angle IBA - \angle IAB \\ &= 180^\circ - \tfrac{1}{2}\angle CBA - \tfrac{1}{2}\angle CAB.\end{aligned}$$

So $\angle DSE = 180^\circ - \frac{1}{2}\angle CBA - \frac{1}{2}\angle CAB$. Let F be the point at which the incircle is tangent to AB. Then $AFIE$ and $BFID$ are both cyclic quadrilaterals (in both cases because of two opposite right angles). Therefore

$$\angle DFE = \angle DFI + \angle IFE = \angle DBI + \angle IAE = \tfrac{1}{2}\angle CBA + \tfrac{1}{2}\angle CAB.$$

We deduce that $\angle DFE + \angle DSE = 180^\circ$, and it follows that S lies on the circumcircle of $\triangle DEF$, which is the incircle of triangle $\triangle ABC$. $\square$

3. We show that if p is the smallest prime divisor of k, then $p-1$ is the smallest k-pable integer. The claim follows from this as $\gcd(p-1,k)=1$.

Note that $1+3+5+\cdots+(2k-1)=k^2$. If ℓ is a k-pable integer, and s is the corresponding sum of the elements of B, then the sum of the elements of A is ℓs. So the sum $1+3+5+\cdots+(2k-1)$ is also equal to $(\ell+1)s$. Hence $(\ell+1)s=k^2$.

Since $\ell+1\geq 2$ we have $\ell+1\geq p$, with p the smallest prime divisor of k. Hence $\ell \geq p-1$. We now show that $\ell=p-1$ is k-pable, from which it will immediately follow that that is the smallest k-pable integer.

First suppose that k is even, so that $p=2$. Then we need to show that we can partition the set $\{1,3,5,\dots,2k-1\}$ into two sets with equal sum of elements. We do this by induction on k. If $k=4$ and $k=6$, we have the partitions $\{1,7\},\{3,5\}$ and $\{1,3,5,9\},\{7,11\}$. If we can partition the set $\{1,3,\dots,2k-1\}$ in such a way, then we can also partition the set $\{1,3,\dots,2(k+4)-1\}$ by taking a partition for the set $\{1,3,\dots,2k-1\}$, and add the elements $2k+1, 2k+7$ to one of the subsets, and $2k+3, 2k+5$ to the other. This completes the induction.

Now suppose that k is odd. Write $k=pm$. Then it suffices to find a subset B of $\{1,3,5,\dots,2k-1\}$ of which the sum of elements is pm^2, since then the sum of the elements of $A=\{1,3,5,\dots,2k-1\}\setminus B$ is $k^2-pm^2 = p^2m^2-pm^2=(p-1)pm^2$, which is exactly $p-1$ times as large as the sum of the elements of B. Consider the subset $B=\{p,3p,\dots,(2m-1)p\}$. Then the sum of elements of B is

$$p+3p+\cdots+(2m-1)p = p\big(1+3+\cdots+(2m-1)\big) = pm^2,$$

as desired. $\square$

4. Substituting $x = 0$ gives $(y+1)f(0) + f(f(y)) = y$, so $f(f(y)) = y \cdot (1 - f(0)) - f(0)$. If $f(0) \neq 1$, the right hand side is a bijective function of y, hence so is the left hand side. So in this case, f is bijective.

We will next show that in case $f(0) = 1$, the function f is bijective as well. So suppose that $f(0) = 1$. Then $f(f(y)) = -1$ for all $y \in \mathbb{R}$. Substituting $y = 0$ gives $f(x) + f(x + f(x)) = 0$, so $f(x + f(x)) = -f(x)$. Substituting $x = f(z)$ and $y = z$ and replacing each occurrence of $f(f(z))$ by -1, we see that

$$(z+1) \cdot -1 + f\big(f(z)f(z) + f(z + f(z))\big) = z,$$

so using that $f(z + f(z)) = -f(z)$, we see that

$$f\big(f(z)^2 - f(z)\big) = 2z + 1.$$

Hence f is surjective. If there exist a and b with $f(a) = f(b)$, then substituting $z = a$ and $z = b$ in the last equation above gives equal left hand sides, therefore $2a + 1 = 2b + 1$, from which it follows that $a = b$. Hence f is injective, and therefore bijective.

So from now on, we may assume that f is bijective, dropping the assumption that $f(0) = 1$. Note that $f(f(y)) = y \cdot (1 - f(0)) - f(0)$ so substituting $y = -1$ gives $f(f(-1)) = -1$. Substituting $y = -1$ in the original equation gives

$$f\big(xf(-1) + f(x-1)\big) = -1 = f(f(-1)).$$

As f is injective, it follows that $xf(-1) + f(x-1) = f(-1)$, so $f(x-1) = f(-1) \cdot (1 - x)$. Substituting $x = z + 1$ then gives $f(z) = -f(-1)z$ for all $z \in \mathbb{R}$. So the function f must be of the form $f(x) = cx$ for all $x \in \mathbb{R}$, with $c \in \mathbb{R}$ a constant. Let us check functions of this form.

Note that

$$\begin{aligned}(y+1)f(x) + f\big(xf(y) + f(x+y)\big) &= (y+1)cx + c(xcy + cx + cy) \\ &= cxy + cx + c^2xy + c^2x + c^2y.\end{aligned}$$

This must equal y for all $x, y \in \mathbb{R}$. Substituting $y = 0$ and $x = 1$ gives $c + c^2 = 0$, so either $c = 0$ or $c = -1$. Substituting $x = 0$ and $y = 1$ gives $c^2 = 1$, so $c = 1$ or $c = -1$. We deduce that $c = -1$, and we see that the function given by $f(x) = -x$ for all $x \in \mathbb{R}$ indeed satisfies the required equation. So the only solution of the equation is the function given by $f(x) = -x$ for all $x \in \mathbb{R}$. $\square$

IMO Team Selection Test 3, June 2017

Problems

1. A circle ω with diameter AK is given. The point M lies in the interior of the circle, but not on AK. The line AM intersects ω in A and Q. The tangent to ω at Q intersects the line through M perpendicular to AK, at P. The point L lies on ω, and is such that PL is tangent to ω and $L \neq Q$. Show that K, L, and M are collinear.

2. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers such that $a_1 + \cdots + a_n = 0$ and define $b_i = a_1 + \cdots + a_i$ for $1 \leq i \leq n$. Suppose that $b_i(a_{j+1} - a_{i+1}) \geq 0$ for all $1 \leq i \leq j \leq n-1$. Show that

$$\max_{1 \leq \ell \leq n} |a_\ell| \geq \max_{1 \leq m \leq n} |b_m|.$$

3. Compute the product of all positive integers n for which $3(n!+1)$ is divisible by $2n-5$.

4. Let $n \geq 2$ be an integer. Find the smallest positive integer m for which the following holds: given n points in the plane, no three on a line, there are m lines such that no line passes through any of the given points, and for all points $X \neq Y$ there is a line with respect to which X and Y lie on opposite sides.

Solutions

1. Let O be the centre of ω, and let V be the intersection of MP and AK. We first show that $\angle PVL = \angle POL$. If V and O coincide, then there is nothing to prove. If they don't, then $\angle OVP = 90° = \angle OLP$, so $OVPL$ or $VOPL$ is a cyclic quadrilateral. (In fact, Q also lies on the corresponding circumcircle.) It follows that $\angle PVL = \angle POL$. So in all cases, $\angle MVL = \angle PVL = \angle POL$. Since PL and PQ are tangent to ω, we have $\triangle OQP \cong \triangle OLP$, so $\angle POL = \frac{1}{2}\angle QOL$. By the interior angle theorem applied to ω this angle is equal to $\angle QAL$. Therefore

$$\angle MVL = \angle POL = \angle QAL = \angle MAL,$$

and it follows that $MVAL$ is a cyclic quadrilateral. So $\angle ALM = 180° - \angle AVM = 90°$. Moreover, by Thales's theorem we have $\angle ALK = 90°$, so $\angle ALM = \angle ALK$, in other words, L, M, and K are collinear. $\square$

2. We have $b_n = 0$. Suppose that there exists an $i \leq n-1$ such that $b_i > 0$ and $a_{i+1} \geq 0$. Then from $b_i(a_{j+1} - a_{i+1}) \geq 0$ it follows that $a_{j+1} \geq a_{i+1} \geq 0$, for all $i \leq j \leq n-1$. So $b_n = b_i + a_{i+1} + a_{i+2} + \cdots + a_n \geq b_i > 0$, which is a contradiction. Therefore $b_i > 0$ implies that $a_{i+1} < 0$. Analogously, if $b_i < 0$ then $a_{i+1} > 0$.

Let k now be such that $|b_k| = \max_{1 \leq m \leq n} |b_m|$. We may assume without loss of generality that $b_k > 0$ (by multiplying all a_i by -1 if necessary). If $k = 1$, then $b_k = a_1$, so $|b_k| = |a_1| \leq \max_{1 \leq \ell \leq n} |a_\ell|$, in which case we are done. Now suppose $k > 1$. If $b_{k-1} > 0$, then by the above, we have $a_k < 0$. On the other hand, $a_k = b_k - b_{k-1} \geq 0$ since b_k was maximal, which is a contradiction. Hence $a_k = b_k - b_{k-1} = b_k + |b_{k-1}| \geq b_k$, so $|b_k| \leq |a_k| \leq \max_{1 \leq \ell \leq n} |a_\ell|$, which is what we needed to prove. $\square$

3. The integers $n = 1, 2, 3, 4$ satisfy the given condition, since in these cases, we have that $2n - 5$ is equal to -3, -1, 1, 3, respectively, which divides $3(n! + 1)$. So suppose from now on that $n > 4$ so that $2n - 5 > 3$.

We first show that if n satisfies the given condition, then $2n - 5$ must be a prime. We distinguish two cases. First suppose that $2n - 5$ is not prime, and has a prime divisor $p > 3$. Since $p \neq 2n - 5$ and since $2n - 5$ is odd, it follows that $p \leq \frac{2n-5}{3} < n$. So $p \mid n!$, but then $p \nmid n! + 1$, so $p \nmid 3(n! + 1)$ since $p \neq 3$. Therefore $2n - 5 \nmid 3(n! + 1)$, so n does not satisfy the given condition. Now suppose that $2n - 5$ is not prime, and only has 3 as prime divisor; so $2n - 5$ is a power of 3 which is greater than 3. However, for

$n > 4$ we have $3 \nmid n! + 1$, so $3(n! + 1)$ is divisible by at most one factor 3. So in this case, n cannot satisfy the given condition either.

So for $n > 4$ satisfying the given condition, we must have that $2n - 5$ is a prime number greater than 3. Write $q = 2n - 5$. Then $q \mid n! + 1$, or equivalently, $n! \equiv -1 \mod q$. Moreover, Wilson's theorem states that $(q-1)! \equiv -1 \mod q$. Therefore

$$\begin{aligned} -1 \equiv (2n-6)! &\equiv (2n-6)(2n-7)\cdots(n+1)\cdot n! \\ &\equiv (-1)\cdot(-2)\cdot\cdots\cdot(-n+6)\cdot n! \equiv (-1)^{n-6}\cdot(n-6)!\cdot n! \\ &\equiv (-1)^n \cdot (n-6)! \cdot -1 \qquad \mod q. \end{aligned}$$

So $(n-6)! \equiv (-1)^n \mod q$. Since $n! \equiv -1 \mod q$, we see that $n \cdot (n-1) \cdot \cdots \cdot (n-5) \equiv (-1)^{n-1} \mod q$. Multiplying this by 2^6 gives

$$2n\cdot(2n-2)\cdot(2n-4)\cdot(2n-6)\cdot(2n-8)\cdot(2n-10) \equiv (-1)^{n-1}\cdot 64 \mod q.$$

Modulo $q = 2n - 5$, the left hand side is $5 \cdot 3 \cdot 1 \cdot -1 \cdot -3 \cdot -5 = -225$.

Suppose n is odd. Then $-225 \equiv 64 \mod q$, so $q \mid -225 - 64 = -289 = -17^2$. Hence $q = 17$ and therefore $n = \frac{17+5}{2} = 11$. Now suppose n is even. Then $-225 \equiv -64 \mod q$, so $q \mid -225 + 64 = -161 = -7 \cdot 23$. Hence $q = 7$ or $q = 23$, which gives $n = 6$ or $n = 14$, respectively.

We check these three possibilities. For $n = 11$ and $2n - 5 = 17$, we have

$$\begin{aligned} 11! &= 1\cdot(2\cdot 9)\cdot(3\cdot 6)\cdot(5\cdot 7)\cdot 4\cdot 8\cdot 10\cdot 11 \\ &\equiv 4\cdot 8\cdot 10\cdot 11 = 88\cdot 40 \equiv 3\cdot 6 \equiv 1 \mod 17, \end{aligned}$$

so $n = 11$ doesn't satisfy the condition. For $n = 14$ and $2n - 5 = 23$,

$$\begin{aligned} 14! &= 1\cdot(2\cdot 12)\cdot(3\cdot 8)\cdot(4\cdot 6)\cdot(5\cdot 14)\cdot(7\cdot 10)\cdot 9\cdot 11\cdot 13 \\ &\equiv 9\cdot 11\cdot 13 = 117\cdot 11 \equiv 2\cdot 11 \equiv -1 \mod 23, \end{aligned}$$

so $n = 14$ does satisfy the condition. Finally, for $n = 6$ and $2n - 5 = 7$ we have $6! \equiv -1 \mod 7$ by Wilson's theorem, so $n = 6$ also satisfies the condition.

So the integers n that satisfy the condition are 1, 2, 3, 4, 6, and 14. Their product is 2016. $\square$

4. We prove the smallest such m equals $\frac{n}{2}$ if n is even and $\frac{n+1}{2}$ if n is odd.

Choose the n points in such a way that they all lie on a circle, and denote them by $P_1, P_2, \ldots, P_n$, in the order in which they lie on the circle. The n segments $P_1P_2, P_2P_3, \ldots, P_nP_1$ must all be intersected by at least one line. As every line intersects the circle at most twice, every line intersects at most two of these segments. This shows that the number of lines is at least $\frac{n}{2}$ if n is even and at least $\frac{n+1}{2}$ if n is odd.

We now show this number of lines suffices. First we show that given four distinct points P_1, P_2, and Q_1, Q_2, there always exists a line with respect to which P_1 and P_2 lie on opposite sides, as well as Q_1 and Q_2. Take the line through the midpoints of the segments P_1P_2 and Q_1Q_2. Suppose this line passes through P_1. Then this line passes through P_2 as well, and as no three of the given points lie on a line, this line does not pass through Q_1, nor through Q_2. Hence one can rotate this line a bit around the midpoint of P_1P_2 in such a way that it still intersects the segment Q_1Q_2, and such that neither P_1 nor P_2 lie on it. The three remaining cases are done analogously.

Now if this line still passes through one of the other given points, one can move this line around a tiny bit for it to no longer pass through any of the given points; this is always possible as there are only finitely many points. Now P_1 and P_2, as well as Q_1 and Q_2, lie on opposite sides of this line.

Assume that n is even; if n is odd, we add an arbitrary point that is not on any line through any pair of given points. The lines that we are about to construct, still give a correct example if we remove the extra point. So we need to construct $\frac{n}{2}$ lines. Fix an arbitrary line such that no line through two of the given points is parallel to it; this is possible as there are only finitely many pairs of points, and infinitely many directions to choose from. Translate this line across the plane. This line will meet the given points one by one. So at some point, there are no points on the line, and on both sides of the line there are $\frac{n}{2}$ points. This will be our first line.

The plane is now split into two regions, say the left region, and the right region. We now add lines in such a way that every line creates a new region in the left region, as well as the right one. (We will only consider regions containing at least one of the given points.) To this end, we pick two points in the left region that aren't separated yet by a line, and two points in the right region that aren't separated yet by a line, and pick a line that separates the two points in the left region, as well as the points in the right region; we have seen before that this is possible. This creates a new region in both the left region and the right region, as the line separates pairs of points in both regions that weren't separated before. If at some point, either the left or the right region no longer contains pairs of points that are not separated, then we will only use the points in the remaining region.

After adding $\frac{n}{2} - 1$ lines, we will have, in both the left and the right region, at least $\frac{n}{2}$ regions, each containing at least one given point; so each region has to contain exactly one given point. In total we have used $\frac{n}{2}$ lines. Hence, it is always possible to satisfy the given condition with $\frac{n}{2}$ lines. $\square$

Junior Mathematical Olympiad, October 2016

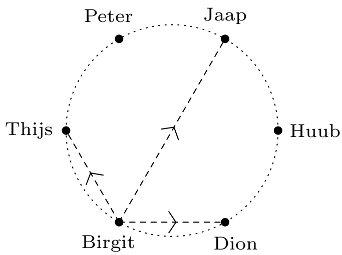
Problems

Part 1

1. The *sum of the digits* of a number is obtained by adding the digits of this number. For example, the sum of the digits of 76 equals $7 + 6 = 13$. The sum of the digits of the double of 76 is $1 + 5 + 2 = 8$.
How many numbers consisting of two digits are there for which the sum of the digits equals the sum of the digits of the double of the number?
Attention: the first digit cannot be a zero. Thus, the number 09*, for example, is ruled out.*

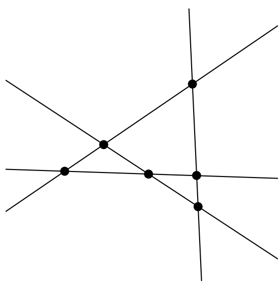
A) 0 B) 8 C) 9 D) 10 E) 11

2. Birgit, Dion, Huub, Jaap, Peter, and Thijs are standing in this order along a circle. They are playing a ball game, in which, at every turn, they pass the ball to a person directly next to them or exactly opposite to them. Initially, Birgit has the ball. After five passes, everybody has had the ball exactly once and the game ends.
Who can have the ball at the end of the game?



A) Only Dion and Thijs
B) Only Dion, Jaap, and Thijs
C) Only Huub, Jaap, and Peter
D) Only Dion, Huub, Peter, and Thijs
E) Everybody except Birgit

3. Four distinct straight lines are drawn on a (infinitely big) piece of paper. The number of points in which two or more lines intersect is counted. In the figure on the right, you see an example in which four lines intersect each other in 6 points. This number of intersection points does not always have to be 6.
What number of intersection points is *not* possible?



A) 1 B) 2 C) 3 D) 4 E) 5

4. A baker's helper is filling cream puffs for one and a half hours. He is not in a hurry and he is filling two cream puffs every minute. At some point, the baker comes in and supervises the helper for a while. This motivates the helper to work a bit faster: during this time he is filling three cream puffs per minute. As soon as the baker has left, the helper reverts to filling the cream puffs at the initial, slow, pace. Afterwards, it turns out that during any continuous one hour period within these one and a half hours, the helper has filled exactly 140 cream puffs.
How many cream puffs are filled by the helper during the full one and a half hours?

A) 180 B) 200 C) 210 D) 230 E) 270

5. Start with the number 60. Then, keep repeating the following two steps:

(1) Throw a die and look at the number that comes up.

(2) If your number is divisible by the number on the die, then you divide your number by the number on the die. If not, then you multiply your number by the number on the die.

In this way, you obtain a sequence of numbers. If your first three rolls are 5, 6, and 3, consecutively, then the first four numbers in your sequence are 60, $60/5 = 12$, $12/6 = 2$, and $2 \times 3 = 6$.
What is the greatest number you can obtain in this way?

A) 60 B) 120 C) 240
D) 360 E) You can obtain arbitrarily large numbers

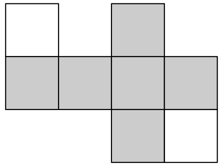
6. Harry and Hermione are trapped in a room in which 6 bottles are put next to each other. From left to right, the bottles are numbered 1 to 6. One of the bottles contains a potion that helps them to escape. On a piece of paper there are four clues to help them:

- 3 of the bottles contain poison, 2 of the bottles contain sleeping draught; the remaining bottle contains the potion to escape.
- Immediately left to the sleeping draught there is poison.
- The smallest bottle contains poison.
- The second bottle from the left and the second bottle from the right have the same content.

Hermione now has enough information to identify the bottle which contains the potion to escape. Which bottle is the smallest bottle?

A) Bottle 1 B) Bottle 2 or 5 C) Bottle 3
D) Botttle 4 E) Bottle 6

7. A *net* of a cube is made by cutting a cube along some of the ribs until you can flatten it out (after cutting, you must still have one connected whole). By doing this in different ways, you can create different nets. The figure on the right consists of 8 squares. The 6 grey squares together form a net.



In how many *other* ways can you choose 6 squares in the figure that together form a net?

A) 3 B) 4 C) 5 D) 7 E) 9

8. Between the digits of the number 2016, we put one or more symbols from $\times$, $+$, and $-$ (you are allowed to use a symbol multiple times). In this way, we can create different numbers, such as $20 + 1 \times 6$, which is 26, and 201×6, which is 1206.
How many of the numbers from 1 to 10 are, just like 26 and 1026, the result of such a calculation? (Attention: you cannot put a $-$ before the 2!)

A) 6 B) 7 C) 8 D) 9 E) 10

Part 2

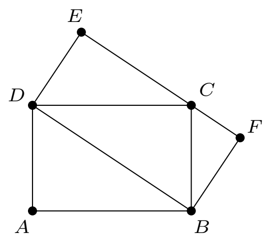
1. All vehicle registration plate numbers in the country Wissewis consist of three two-digit numbers. A plate number is considered beautiful if it has the following two properties:

- it consists of six distinct digits;
- the first number is smaller than the second number and the second number is smaller than the third number.

An example of a beautiful plate number is 03-29-64.
How many beautiful plate numbers are there that have 61 as the first number?

2. Alice, Bob, Carla, Daan, and Eva are standing in this order along a circle (Bob is standing to the left of Alice). Each of them has a number of sweets, they have 100 sweets in total. All at the same time, they give part of their sweets to their left neighbour: Alice gives away $\frac{1}{3}$ of her sweets, Bob $\frac{1}{4}$, Carla $\frac{1}{5}$, Daan $\frac{1}{6}$, and Eva $\frac{1}{7}$. After this, everybody has the same number of sweets as before.
How many sweets does Eva have?

3. In the figure on the right, rectangles $ABCD$ and $BDEF$ are shown. The length of AB is 8 and the length of BC is 5.
What is the area of pentagon $ABFED$?



4. In this problem we consider three-digit numbers of which no digit is a zero. Such a number is called a *lucky number* if:

- the number is divisible by 4, and
- if you change the order of the three digits, you will still always get a number divisible by 4.

For example, the number 132 is not a lucky number, because 132 is divisible by 4, but 231 is not.
How many lucky numbers are there?

5. How many times a day (which is 24 hours) are the small hand and the big hand of the clock perpendicular?

6. Janneke, Karin, Lies, Marieke, and Nadine participated in a running race. They all finished at distinct times except for two of them; they finished at the same time. Moreover, we know that:

- at least three runners finished before Janneke;
- after Karin finished but before Lies finished, exactly two others crossed the finish line;
- Marieke was not the first to finish;
- shortly after Nadine finished, Janneke crossed the finish line; nobody else was in-between.

Which two runners finished at the same time?

7. For all positive integers a and b we make the number $a \heartsuit b$. The following rules hold:

- rule 1: $1 \heartsuit 1 = 1$;
- rule 2: $a \heartsuit b = b \heartsuit a$;
- rule 3: $a \heartsuit (b + c) = a + (a \heartsuit b) + (a \heartsuit c)$.

From these rules it follows, for example, that

$$2 \heartsuit 1 = 1 \heartsuit 2 = 1 \heartsuit (1 + 1) = 1 + 1 \heartsuit 1 + 1 \heartsuit 1 = 1 + 1 + 1 = 3.$$

Calculate $20 \heartsuit 16$.

8. We create a sequence of numbers. To get the next number in the sequence, we repeatedly do the following:

- if the previous number is odd: multiply this number by itself and add 3;
- if the previous number is even: divide this number by 2.

For example, when we start with 5, we obtain $5 \times 5 + 3 = 28$ as second number and $\frac{28}{2} = 14$ as third number in the sequence. As starting number we are allowed to choose any of the numbers from 1 to 1000.
For how many of these starting numbers will the tenth number in the sequence be smaller than 10?

Solutions

Part 1

1. D) 10

2. B) Only Dion, Jaap, and Thijs

3. B) 2

4. B) 200

5. E) You can obtain arbitrarily large numbers

6. E) Bottle 6

7. C) 5

8. B) 7

Part 2

1. 90

2. 28

3. 60

4. 8

5. 44

6. Janneke and Lies

7. 924

8. 17

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