53rd Dutch Mathematical Olympiad 2014

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

The selection process for IMO 2015 started with the first round in January 2014, held at the participating schools. The paper consisted of eight multiple choice questions and four open questions, to be solved within 2 hours. In total 23% more students than in 2013 participated in this first round: to be precise: 9161 students of 323 secondary schools.

The 1000 best students were invited to the second round, which was held in March at twelve universities in the country. This round contained five open questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

The 130 best students were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total about 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

The final round in September contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 53rd edition 2014.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad 2014 were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. Also, they worked on weekly problem sets under supervision of a personal trainer.

Among the participants of the training programme, there were some extra girls, as this year we participated again in the European Girls' Mathematical Olympiad (EGMO). In total there were eight girls competing to be in the EGMO team. The team of four girls was selected by a selection test, held on 6 March 2015. They attended the EGMO in Minsk, Belarus from 14 until 20 April, and the team returned with two honourable mentions. For more information about the EGMO (including the 2015 paper), see www.egmo.org.

The same selection test was used to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Mersch, Luxembourg, from 8 until 10 May. The Dutch team received four bronze medals and three silver medals. For more information about the BxMO (including the 2015 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2015 was selected by two team selection tests on 5 and 6 June 2015. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Chiang Mai, from 30 June until 8 July.

For younger students the Junior Mathematical Olympiad was held in October 2014 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2015 in Thailand consists of

- Eva van Ammers (17 years old)
  - participated in EGMO 2014, honourable mention at EGMO 2015
- Dirk van Bree (17 years old)
  - bronze medal at BxMO 2015
- Tim Brouwer (18 years old)
  - bronze medal at BxMO 2015
- Yuhui Cheng (19 years old)
  - participated in BxMO 2014, bronze medal at BxMO 2015
- Mike Daas (17 years old)
  - bronze medal at BxMO 2015
- Bob Zwetsloot (17 years old)
  - bronze medal at BxMO 2013, bronze medal at BxMO 2014, silver medal at BxMO 2015
  - observer C at IMO 2014

We bring as observer C the promising young student

- Levi van de Pol (13 years old)
  - silver medal at BxMO 2015

The team is coached by

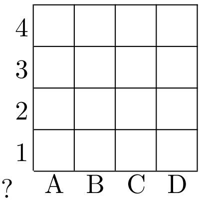
- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Merlijn Staps (observer B), Utrecht University

First Round, January 2014

Problems

A-problems

1. We are given a 4×4 table and want to colour four of the 16 cells black. This should be done in such a way that every row and every column has exactly one black cell, and no two black cells are diagonally adjacent (share a corner point).
In how many ways can we choose the four black cells?



A) 1 B) 2 C) 3 D) 4 E) It is impossible.

2. A pond contains both red and yellow carp. Two fifths of the carp are yellow, the rest of the carp are red. Three quarters of the yellow carp are female. In total, there are an equal number of male and female carp. Which fraction of the total carp population are red males?

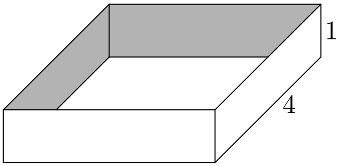
A) $\frac{1}{5}$ B) $\frac{1}{4}$ C) $\frac{3}{10}$ D) $\frac{2}{5}$ E) $\frac{1}{2}$

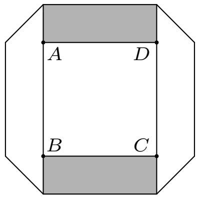
3. Seven lily pads are numbered 1 through 7 from left to right. A frog jumps along these pads. It can jump to the left and to the right, but only by leaps of three or five pads at once. For example, starting from pad 2, it can only leap to pads 5 and 7. The frog wants to make a journey in which he visits each pad exactly once (so the first and last pad on his journey will be different).

Which pads can be the starting point of such a journey?

A) pads 1 to 7 B) pads 1, 3, 5, and 7 C) pads 3 and 5
D) pad 4 E) none of the pads

4. A square paper ring has height 1, and the sides have length 4. The ring is depicted in the left hand figure. By folding it flat on the tabletop, we get the right hand figure, where $ABCD$ is a square.



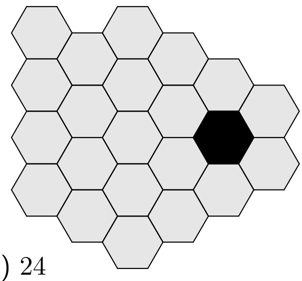


What is the length of side AB?

A) $\frac{5}{2}$ B) 3 C) $\frac{7}{2}$ D) 4 E) $\frac{9}{2}$

5. Tom and Jerry were running a race. The number of runners finishing before Tom was equal to the number of runners finishing after him. The number of runners finishing before Jerry was three times the number of runners finishing after him. In the final ranking, there are precisely 10 runners in between Tom and Jerry. All runners finished the race, and no two runners finished at the same time.
How many runners participated in the race?

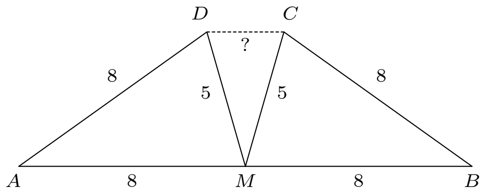
A) 22 B) 23 C) 41 D) 43 E) 45



6. A garden with a pond (the black hexagon) will be tiled using hexagonal tiles as in the figure. The tiles come in three colours: red, green and blue. No two tiles that share a side can be of the same colour.
In how many ways can the garden be tiled?

A) 3 B) 6 C) 12 D) 18 E) 24

7. In the figure, a quadrilateral $ABCD$ is drawn. The midpoint of side AB is called M. The four line segments AM, BM, BC, and AD each have length 8, and the line segments DM and CM both have length 5.
What is the length of line segment CD?



Beware: the figure is not drawn to scale.

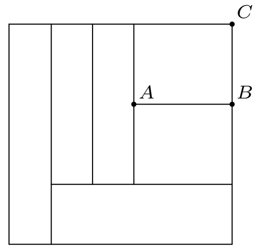
A) 3 B) $\frac{40}{13}$ C) $\frac{25}{8}$ D) $\frac{16}{5}$ E) $\frac{13}{4}$

8. A motorboat is moving with a speed of 25 kilometres per hour, relative to the water. It is going from Arnhem to Zwolle, moving with the constant current. At a certain moment, it has travelled 42% of the total distance. From that point on, it takes the same amount of time to reach Zwolle as it would to travel back to Arnhem.
What is the speed of the current (in kilometres per hour)?

A) 3 B) 4 C) $\frac{9}{2}$ D) 5 E) 6

B-problems

The answer to each B-problem is a number.



1. A square is divided into six rectangles, all of the same area. The length of side AB equals 5.
What is the length of side BC?
Beware: the figure is not drawn to scale.

2. Carl has a large number of apples and pears. He wants to pick ten pieces of fruit and place them in a row. He wants to do it in such a way that there is no pear anywhere between two apples. For example, the fruit sequences AAAAAAAAAA and AAPPPPPPPP are allowed, but AAPPPPPPPA and APPPPPPPAA are not.
How many sequences can Carl make?

3. If you were to compute

$$\underbrace{999\ldots99}_{\text{2014 nines}} \times \underbrace{444\ldots44}_{\text{2014 fours}}$$

and then add up all digits of the resulting number, what number would the final outcome be?

4. We consider 5×5-tables containing a number in each of the 25 cells. The same number may occur in different cells, but no row or column contains five equal numbers. Such a table is called *pretty* if in each row the cell in the middle contains the average of the numbers in that row, and in each column the cell in the middle contains the average of the numbers in that column. The *score* of a pretty table is the number of cells that contain a number that is smaller than the number in the cell in the very middle of the table.
What is the smallest possible score of a pretty table?

Solutions

A-problems

A1. B) 2 Suppose that we colour B2 black. Then the surrounding 8 cells cannot be coloured black. Indeed, the cells above, below, to the left, and to the right of B2 are in the same row or column as B2, while the other four cells are diagonally adjacent to B2. This way, only row 4 and column D remain and in each we can colour only one cell black. In total we can colour no more than 3 cells black. We conclude that B2 cannot be coloured black.

Similarly, we may deduce that cells B3, C2, and C3 cannot be coloured black. It follows that in row 2, we can only colour A2 or D2 black. If we colour A2 black, then in row 3 cell D3 must be coloured black because A3 is in the same column as A2. In rows 1 and 4 we now have no choice but to colour cells C1 and B4 black. This gives us one solution.

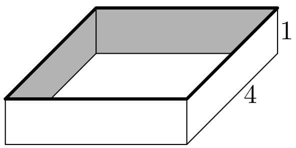
If instead of A2 we colour cell D2 black, then we find a solution where cells D2, A3, B1, and C4 are coloured black. In total, we have two ways of choosing the black squares.

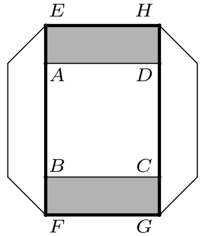
A2. D) $\frac{2}{5}$ From the given data, we deduce that $\frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$ of the carp are yellow females. Since half the carp are female, we find that $\frac{1}{2} - \frac{3}{10} = \frac{1}{5}$ of the carp are red females. Finally, using the fact that three fifths of the carp are red, we see that $\frac{3}{5} - \frac{1}{5} = \frac{2}{5}$ of the carp are red males.

A3. C) 3 and 5 The journey that visits the pads in the order 3, 6, 1, 4, 7, 2, and 5 (or in the opposite order), shows that pads 3 and 5 are the starting pad of a possible journey. We will see that these are the only possible starting points.

We say that two pads are *neighbours* if the frog can jump from one pad to the other (and hence also the other way around). Every intermediate pad in the frog's journey must have at least two neighbours: the pad the frog came from and the pad it will go next. Since pads 3 and 5 have only one neighbour (pad 6 and pad 2 respectively), these must be the starting point and end point of the frog's journey. The other pads must therefore be intermediate pads.

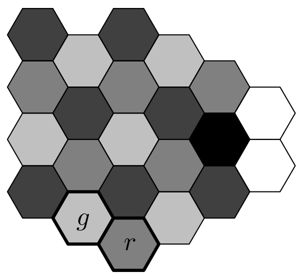
A4. B) 3 Consider the top rim of the paper ring (indicated in bold). The rim has length $4 \times 4 = 16$. In the folded state, the rim becomes rectangle $EFGH$. Since $|AE| = |BF| = |CG| = |DH| = 1$, we find that $|AB| + |FG| + |CD| + |EH| = 16 - 4 = 12$. These four lengths equal the length of the sides of square $ABCD$. It follows that the square has sides of length $\frac{12}{4} = 3$.





A5. E) 45 Let n be the number of runners. The number of runners that finished before Tom equals $\frac{1}{2}(n-1)$ (half of all runners besides Tom). The number of runners that finished before Jerry equals $\frac{3}{4}(n-1)$. Since exactly 10 runners finished between Tom and Jerry, it follows that $\frac{3}{4}(n-1) - \frac{1}{2}(n-1)$ equals 11 (Tom and the 10 runners between Tom and Jerry). We find that $\frac{1}{4}(n-1) = 11$, hence $n = 4 \times 11 + 1 = 45$. There were 45 runners participating in the race.

A6. D) 18 We start by colouring the two indicated tiles at the bottom. This can be done in six ways: there are three options for the first tile and for each option there are two possible colours for the second tile. In the figure, the colours red (r) and green (g) are chosen.



Now that these two tiles are coloured, the colours of most of the other tiles are determined as well. The tile above the red tile can only be blue. The tile above the green tile must be red and therefore the tile left of the green tile must be blue. In this way the colours of all tiles, except the two on the right (white in the figure), are fixed. For these last two tiles, there are three possible colourings. The upper and lower tile can be coloured either green and red, or blue and red, or blue and green.

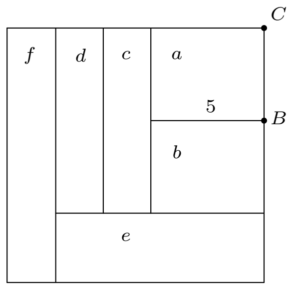
Since each of the six allowed colourings of the first two tiles can be extended in three ways to a complete colouring, we find a total of $6 \times 3 = 18$ possible colourings.

A7. C) $\frac{25}{8}$ Observe that AMB is a straight angle. This implies that $\angle AMD + \angle DMC + \angle CMB = 180°$. Since triangles AMD and BMC are equal (three equal sides), we see that $\angle CMB = \angle MDA$. Hence $\angle DMC = 180° - \angle AMD - \angle MDA = \angle DAM$, because the angles of triangle AMD sum to 180 degrees. It follows that DMC and DAM are isosceles triangles with equal apex angles. Hence these two triangles are equal up to scaling. This means that $\frac{|CD|}{|DM|} = \frac{|DM|}{|AD|}$. Therefore, the length of CD equals $\frac{5}{8} \cdot 5 = \frac{25}{8}$.

A8. B) 4 From the mentioned point, it takes the same time to go 42% of the distance upstream and to go 58% of the distance downstream. This means that the boat is $\frac{58}{42}$ times as fast going downstream as going upstream. If the water flows at a speed of v kilometres per hour, then we find $\frac{25+v}{25-v} = \frac{58}{42}$. Hence $58 \cdot (25-v) = 42 \cdot (25+v)$, or $1450 - 58v = 1050 + 42v$. We find $400 = 100v$, hence $v = 4$.

B-problems

B1. $\frac{24}{5}$ The six rectangles have equal areas. Rectangles c and d are twice as tall as rectangle a and therefore also twice as thin. Hence they have width $\frac{5}{2}$. Rectangle e thus has a width of $\frac{5}{2} + \frac{5}{2} + 5 = 10$ and must be half as tall as rectangle a. This means that rectangle f is precisely $\frac{5}{2}$ times as tall as rectangle a and therefore has a width of $\frac{5}{5/2} = 2$. It follows that the square has sides of length $5 + \frac{5}{2} + \frac{5}{2} + 2 = 12$. Because the square has a height of $\frac{5}{2}$ times the height of rectangle a, the height of rectangle a equals $|BC| = \frac{12}{5/2} = \frac{24}{5}$.



B2. 56 One sequence consists of pears alone. Next, we count sequences containing at least one apple. In such a sequence, all apples occur consecutively, because there can be no pear anywhere between two apples. If we want to have 8 apples, we can place them in positions 1 through 8, 2 through 9, or 3 through 10. This gives three possible sequences. In this way we find 1 sequence containing 10 apples, 2 sequences containing 9 apples, 3 sequences containing 8 apples, and so on, ending with 10 sequences containing 1 apple. In total there are $1 + 2 + 3 + \cdots + 10 = 55$ sequences containing at least one apple. The total number of sequences is therefore $55 + 1 = 56$.

B3. 18126 A good strategy is to first consider smaller examples. We find

$$\begin{aligned} 9 \times 4 &= 40 - 4 &&= 36, \\ 99 \times 44 &= 4400 - 44 &&= 4356, \\ 999 \times 444 &= 444000 - 444 &&= 443556, \\ 9999 \times 4444 &= 44440000 - 4444 &&= 44435556. \end{aligned}$$

The pattern should be clear. To solve the problem, observe that $999\ldots 99 = 1000\ldots 00 - 1$, where the first number has 2014 zeroes. The product is therefore equal to

$$\underbrace{444\ldots 44}_{\text{2014 fours}}\underbrace{000\ldots 00}_{\text{2014 zeroes}} - \underbrace{444\ldots 44}_{\text{2014 fours}} = \underbrace{444\ldots 44}_{\text{2013 fours}}3\underbrace{555\ldots 55}_{\text{2013 fives}}6.$$

Adding these digits, we obtain $2013 \cdot 4 + 3 + 2013 \cdot 5 + 6 = 2013 \cdot 9 + 9 = 18126$.

B4. 3 We first show that every pretty table has a score of at least 3. Consider such a table and let a be the number at the very middle. The five numbers in the middle row have an average of a and are not all equal to a. Hence at least one of these numbers must be smaller than a. Similarly, at least one of the numbers in the middle column must be smaller than a. Let this number be b. Since b is the average of the numbers in its row, one of the numbers in that row must be smaller than b, and hence also smaller than a. Thus the table contains at least three different cells that have a number smaller than the number in the very middle. Its score is therefore at least 3.

4	4	3	4	0
4	4	3	4	0
3	3	0	3	−9
4	4	3	4	0
0	0	−9	0	−36

In the figure on the right you can find a pretty table with a score equal to 3. It follows that 3 is the smallest possible score.

Second Round, March 2014

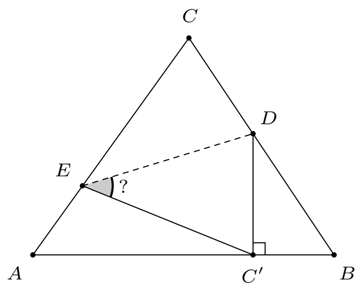
Problems

B-problems

The answer to each B-problem is a number.

B1. Brenda is filling pouches from an unlimited supply of red and blue marbles. In each pouch she puts more red than blue marbles, and each pouch can contain at most 25 marbles. For example, she can make a pouch containing 6 red marbles and 2 blue marbles, or a pouch containing 1 red marble and 0 blue marbles.
How many differently filled pouches can she make in total?

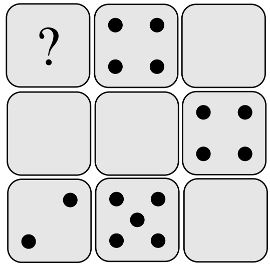
B2. In the figure an equilateral triangle ABC is drawn with points D and E on sides BC and AC. When folding the triangle along the line DE, the vertex C is folded onto point C' on line AB. Furthermore, $\angle DC'B = 90°$ holds.
What is the size of $\angle DEC'$?
Beware: the figure is not drawn to scale.



B3. For how many of the integers n from 1 up to and including 100 is the number $8n + 1$ a perfect square?

B4. Evan and nine other people are standing in a circle. All ten of them think of an integer (that may be negative) and whisper their number to both of their neighbours. Afterwards, they all state the average of the two numbers that were whispered in their ear. Evan states the number 10, his right neighbour states the number 9, the next person along the circle states the number 8, and so on, finishing with Evan's left neighbour who states the number 1.
What number did Evan have in mind?

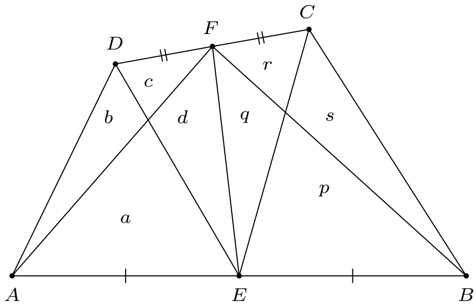
B5. The numbers of dots on two opposite faces of a die always sum to 7. Nine identical dice are glued in a 3×3-array. This is done in such a way that when two faces are glued together, they must contain the same number of dots. In the figure you can see the top view of the array. For five of the dice the number of dots is not shown.
What number of dots must be on the place of the question mark?



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. We are given a quadrilateral $ABCD$. The midpoint of AB is denoted by E and the midpoint of CD is denoted by F. The segments AF, BF, CE, DE, and EF divide the quadrilateral into eight triangles. The areas of these triangles are denoted by the letters from a to d and p to s, see the figure.



(a) Prove that $a+d = p+q$.

(b) Prove that $a+r = c+p$.

(c) Prove that $b+s = d+q$.

C2. A positive integer n is called a *jackpot number* if it has the following property: there exists a positive integer k consisting of two or more digits, all of which are equal (such as 11111 or 888), for which the product $n \cdot k$ is again a number consisting of equal digits. For example, 3 is a jackpot number because $3 \cdot 222 = 666$.

(a) Find a jackpot number consisting of 10 digits and prove that it is a jackpot number.

(b) Prove that 11 is not a jackpot number.

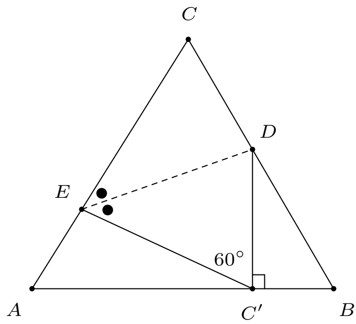
(c) Determine whether 143 is a jackpot number and prove that your answer is correct.

Solutions

B-problems

B1. 169 Brenda can make 25 differently filled pouches without blue marbles as she can put 1 to 25 red marbles in a pouch. There are 23 differently filled pouches possible containing 1 blue marble because 2 to 24 red marbles may be added. Using 2 blue marbles there are 21 possibilities, namely by adding 3 to 23 red marbles. In total, there are $25+23+\ldots+1 = 169$ differently filled pouches that Brenda can make.

B2. $45°$ First, we notice that triangles CDE and $C'DE$ are each other's mirror image and hence have equally sized angles. In particular, we have $\angle DC'E = \angle DCE = 60°$. Furthermore, we have $\angle CED = \angle DEC'$, see the figure.



From $\angle AC'B = 180°$ it follows that $\angle AC'E = 180° - 90° - 60° = 30°$. The sum of the angles in triangle $AC'E$ is $180°$ and hence we find $\angle AEC' = 180° - 60° - 30° = 90°$.
From $\angle AEC = 180°$ it follows that $\angle CEC' = 180° - 90° = 90°$. We conclude that $\angle DEC' = \frac{1}{2}\angle CEC' = 45°$.

B3. 13 If $8n+1$ is the square of an integer, then this integer must be odd. Conversely, the square of an odd integer is always a multiple of 8 plus 1. Indeed, suppose that k is an odd integer, then we may write $k = 2\ell + 1$ for an integer ℓ. We see that

$$k^2 = (2\ell+1)^2 = 4\ell^2 + 4\ell + 1 = 4\ell(\ell+1) + 1.$$

Because either ℓ or $\ell+1$ is even, we deduce that $4\ell(\ell+1)$ is divisible by 8. Hence, k^2 is a multiple of 8 plus 1.

As a result, we only need to determine the number of odd squares x for which $8 \cdot 1 + 1 \leq x \leq 8 \cdot 100 + 1$. These are the squares $3^2 = 9$, $5^2 = 25$ up to and including $27^2 = 729$, because $29^2 = 841$ is greater than 801. Thus, the number of squares of the desired form is 13.

B4. 5 The number that Evan came up with is denoted by c_{10}, the number of his right neighbour is denoted by c_9, continuing in this way until the left neighbour of Evan, whose number is denoted by c_1. From the data we deduce that

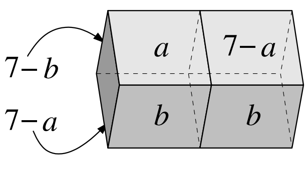
$$\begin{aligned} c_{10} + c_8 &= 2 \cdot 9 = 18, \\ c_8 + c_6 &= 2 \cdot 7 = 14, \\ c_6 + c_4 &= 2 \cdot 5 = 10, \\ c_4 + c_2 &= 2 \cdot 3 = 6, \\ c_2 + c_{10} &= 2 \cdot 1 = 2. \end{aligned}$$

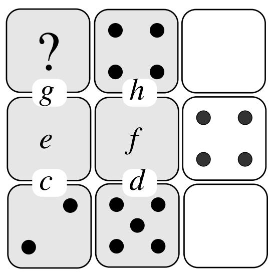
Adding up these equations yields $2(c_2 + c_4 + c_6 + c_8 + c_{10}) = 50$, hence $c_2 + c_4 + c_6 + c_8 + c_{10} = 25$.
Finally, we find $c_{10} = (c_2 + c_4 + c_6 + c_8 + c_{10}) - (c_2 + c_4) - (c_6 + c_8) = 25 - 6 - 14 = 5$.

B5. 3 The numbers of dots on opposite faces of a die will be called *complementary*. Together, they always add to 7. Consider a pair of dice that touch in faces with equal numbers of dots. We still allow them to rotate with respect to each other. When rotating the dice, their faces show the same numbers of dots, but in reverse cyclic order.

Consider the situation where the numbers of dots on the top faces of the dice are complementary, say a and $7 - a$. This is depicted in the figure on the left. On the four faces around the gluing axis, the left hand die will have a, b, $7 - a$, and $7 - b$ dots in this order, for some b. Hence, the right hand die has these numers in reverse cyclic order: $7 - a$, b, a, and $7 - b$ dots on the corresponding faces. It follows that the dice have the same number of dots on the front face, namely b, and the same number of dots on the back face, namely $7 - b$.

Conversely, if two glued dice have the same number of dots on the front faces (or back faces), then the numbers of dots on the top faces must be complementary.





We will now apply this to the six dice in the first two columns of the 3×3 array, see the figure on the right. The two dice in the third row show complementary numbers of dots on their top faces, namely 2 and 5. Therefore, the numbers of dots c and d, on the faces where the dice are glued to dice in the second row, must be equal. This in turn implies that the numbers of dots e and f on the top faces must be complementary. Therefore, the numbers g and h are equal. Finally, the top faces of the dice in the first row must show complementary numbers of dots. Hence, there must be 3 dots on the place of the question mark.

C-problems

C1. (a) First, we notice that $a+d$ is the area of triangle AEF and $p+q$ is the area of triangle BEF. The base AE of triangle AEF has the same length as the base BE of triangle BEF. Because the two triangles have the same height, they must also have equal areas.

(b) Here we use that triangles DEF and CEF have bases of the same length ($|DF| = |CF|$), and equal corresponding heights. Hence, they have equal areas. This implies that $c + d = q + r$. Subtracting the equation of part (a) yields $c - a = r - p$, or $a + r = c + p$, as required.

(c) The heights of triangles AED, BEC and ABF with respect to the bases AE, BE, and AB are denoted by x, y, and z. Because F is the midpoint of CD, the height z is the average of the heights x and y, in formulas: $\frac{x+y}{2} = z$. The area of triangle AED is $\frac{1}{2} \cdot x \cdot |AE|$, the area of triangle BEC is $\frac{1}{2} \cdot y \cdot |BE|$, and the area of triangle ABF is $\frac{1}{2} \cdot z \cdot |AB|$. Because E is the midpoint of AB we have $|AE| = |BE|$ and $|AB| = 2 \cdot |AE|$. The sum of the areas of triangles AED and BEC is thus equal to $\frac{1}{2} \cdot (x + y) \cdot |AE| = z \cdot |AE|$, while the area of triangle ABF is equal to $\frac{1}{2} \cdot z \cdot 2|AE| = z \cdot |AE|$. Hence, these areas are equal and we find $a + b + p + s = a + d + p + q$. By subtracting $a + p$ on both sides of the equation, we find $b + s = d + q$, as required.

C2. (a) Consider the ten digit number 1001001001. If we multiply it with 111, we get the number 111111111111 which consists of ones only. Hence, the number 1001001001 is a jackpot number.

(b) Let k be a number of at least two digits, all of which are equal, say equal to a. Remark that $a \neq 0$. We have to prove that the digits of the number

$$11k \;=\; k + 10k \;=\; a \cdots a \;+\; a \cdots a0$$

are not all equal.

The last digit of $11k$ is a. We will show that the second last digit of $11k$ is unequal to a. There are two cases. If $a \leq 4$, then the second last digit of $11k$ is equal to $a + a$. This is unequal to a as $a \neq 0$. If $a \geq 5$, then the second last digit of $11k$ is equal to $a + a - 10$. This is unequal to a as $a \neq 10$. We conclude that 11 is not a jackpot number.

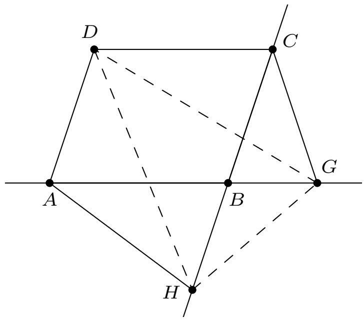
(c) That 143 is a jackpot number follows directly from the fact that $143 \cdot 777 = 111111$.

Final Round, September 2014

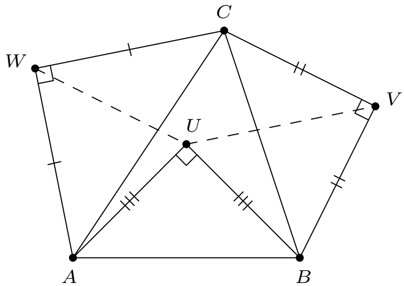
Problems

1. Determine all triples (a, b, c), where a, b, and c are positive integers that satisfy $a \le b \le c$ and $abc = 2(a + b + c)$.

2. Version for junior students
Let $ABCD$ be a parallelogram with an acute angle at A. Let G be a point on the line AB, distinct from B, such that $|CG| = |CB|$. Let H be a point on the line BC, distinct from B, such that $|AB| = |AH|$.
Prove that triangle DGH is isosceles.



2. Version senior students
On the sides of triangle ABC, isosceles right-angled triangles AUB, CVB, and AWC are placed. These three triangles have their right angles at vertices U, V, and W, respectively. Triangle AUB lies completely inside triangle ABC and triangles CVB and AWC lie completely outside ABC. See the figure.
Prove that quadrilateral $UVCW$ is a parallelogram.



3. At a volleyball tournament, each team plays exactly once against each other team. Each game has a winning team, which gets 1 point. The losing team gets 0 points. Draws do not occur. In the final ranking, only one team turns out to have the least number of points (so there is no shared last place). Moreover, each team, except for the team having the least number of points, lost exactly one game against a team that got less points in the final ranking.

a) Prove that the number of teams cannot be equal to 6.

b) Show, by providing an example, that the number of teams could be equal to 7.

4. A quadruple (p, a, b, c) of positive integers is called a *Leiden quadruple* if

- p is an odd prime number,
- a, b, and c are distinct and
- $ab + 1$, $bc + 1$ and $ca + 1$ are divisible by p.

a) Prove that for every Leiden quadruple (p, a, b, c) we have $p + 2 \leq \frac{a+b+c}{3}$.

b) Determine all numbers p for which a Leiden quadruple (p, a, b, c) exists with $p + 2 = \frac{a+b+c}{3}$.

5. We consider the ways to divide a 1 by 1 square into rectangles (of which the sides are parallel to those of the square). All rectangles must have the same *circumference*, but not necessarily the same shape.

a) Is it possible to divide the square into 20 rectangles, each having a circumference of 2.5?

b) Is it possible to divide the square into 30 rectangles, each having a circumference of 2?

Solutions

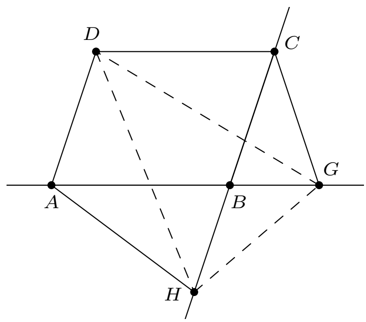
1. Suppose that (a,b,c) is a solution. From $a \leq b \leq c$ it follows that $abc = 2(a+b+c) \leq 6c$. Dividing by c yields $ab \leq 6$. We see that $a = 1$ or $a = 2$, because from $a \geq 3$ it would follow that $ab \geq a^2 \geq 9$.

We first consider the case $a = 2$.
From $ab \leq 6$ it follows that $b = 2$ or $b = 3$. In the former case, the equation $abc = 2(a+b+c)$ yields $4c = 8 + 2c$ and hence $c = 4$. It is easy to check that the triple $(2,2,4)$ we got is indeed a solution. In the latter case, we have $6c = 10 + 2c$, hence $c = \frac{5}{2}$. Because c has to be an integer, this does not give rise to a solution.

Now we consider the case $a = 1$.
We get that $bc = 2(1 + b + c)$. We can rewrite this equation to obtain $(b-2)(c-2) = 6$. Remark that $b - 2$ cannot be negative (and hence also $c - 2$ cannot be negative). Otherwise, we would have $b = 1$, yielding $(1-2)(c-2) = 6$, from which it would follow that $c = -4$. However, c has to be positive.
There are only two ways to write 6 as a product of two non-negative integers, namely $6 = 1 \times 6$ and $6 = 2 \times 3$. This gives rise to two possibilities: $b - 2 = 1$ and $c - 2 = 6$, or $b - 2 = 2$ and $c - 2 = 3$. It is easy to check that the corresponding triples $(1,3,8)$ and $(1,4,5)$ are indeed solutions.

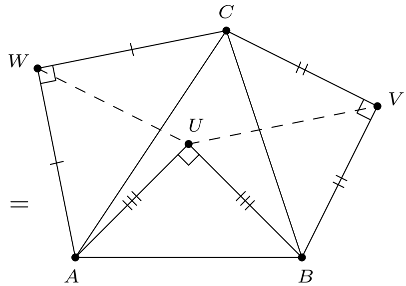
Thus, the only solutions are $(2,2,4)$, $(1,3,8)$, and $(1,4,5)$. $\square$

2. Version for junior students



We know that $\angle ABH = \angle CBG$, because these are opposite angles. Because triangles ABH and CBG are isosceles, we have $\angle AHB = \angle ABH$ and $\angle CBG = \angle CGB$. Triangles ABH and CBG are similar (AA) and hence we have $\angle BAH = \angle BCG$. Because $ABCD$ is a parallelogram, we have $\angle DAB = \angle DCB$ and hence $\angle DAH = \angle DAB + \angle BAH = \angle DCB + \angle BCG = \angle DCG$ holds. Because $ABCD$ is a parallelogram, we have $|CD| = |AB| = |AH|$ and $|AD| = |BC| = |CG|$. Therefore, triangles DAH and GCD are congruent (SAS) and we have $|DH| = |DG|$. In other words, triangle DGH is isosceles. $\square$

2. Version for senior students



Because triangle AUB is isosceles with top angle $\angle AUB = 90°$, we have $\angle UAB = 45°$. In the same way, we have $\angle CAW = 45°$. Combining these two equalities, we find $\angle WAU = 45° + \angle CAU = \angle CAB$. By the Pythagorean theorem, we find $2|AW|^2 = |AW|^2+|CW|^2 = |AC|^2$ and hence $|AW| = \frac{1}{2}\sqrt{2}\cdot|AC|$. In the same way we find $|AU| = \frac{1}{2}\sqrt{2}\cdot|AB|$. Hence, triangles WAU and CAB are similar (SAS) with scale factor $\frac{|AW|}{|AC|} = \frac{1}{2}\sqrt{2} = \frac{|AU|}{|AB|}$. In particular, we find $|WU| = \frac{1}{2}\sqrt{2}\cdot|BC| = |CV|$.

In the same way, we see that triangles VBU and CBA are similar and that $|VU| = \frac{1}{2}\sqrt{2}\cdot|AC| = |CW|$. It follows that in quadrilateral $UVCW$ the opposite sides have equal lengths, hence $UVCW$ is a parallelogram. □

3. a) Suppose that the number of teams is 6. We shall derive a contradiction.

First remark that the number of games equals $\frac{6\times5}{2} = 15$. Hence, the total number of points also equals 15.

Let team A be the (only) team with the lowest score. Team A has *at most* 1 point, because if team A had 2 or more points, then each of the other five teams would have at least 3 points, giving a total number of points that is at least $2+3+3+3+3+3 = 17$. Each team on the second last place in the ranking has lost to team A, because this is the only team with a lower score. Hence, team A also has *at least* 1 point. We deduce that A has exactly 1 point and that there is exactly one team, say team B, in the second last place in the ranking.

Team B has at least 2 points and the remaining four teams, teams C, D, E and F, each have at least 3 points. The six teams together have at least $1 + 2 + 3 + 3 + 3 + 3 = 15$ points. If team B had more than 2 points, or if any of the teams C through F had more than 3 points, then the total number of points would be greater than 15, which is impossible. Hence, team B has exactly 2 points and teams C through F each have exactly 3 points. The four teams C through F each lost to a team having a lower score (team A or team B). Hence, together, team A and team B must have won at least 4 games. This contradicts the fact that together they have only $1 + 2 = 3$ points.

□

b) In the table below there is a possible outcome for 7 teams called A through G. In the row corresponding to a team, crosses indicate wins against other teams. Row 2, for example, indicates that team B won against teams C and D and obtained a total score of 2 points. Each team (except A) has indeed lost exactly one match against a team with a lower score. These matches are indicated in bold.

	A	B	C	D	E	F	G	Score
A	-	**X**						1
B		-	**X**	**X**				2
C	X		-		**X**		**X**	3
D	X		X	-		**X**		3
E	X	X		X	-	X		4
F	X	X	X			-	X	4
G	X	X		X	X		-	4

$\square$

4. a) Without loss of generality, we may assume that $a < b < c$. The integers a and c are not divisible by p because that would imply that $ac+1$ is a multiple of p plus 1, hence not divisible by p. Since $bc+1$ and $ac+1$ are both divisible by p, their difference $(bc+1)-(ac+1) = (b-a)c$ is divisible by p as well. Hence, since c is not divisible by p, it must be the case that $b-a$ is divisible by p. Similarly, $(ac+1)-(ab+1) = a(c-b)$ is divisible by p and since a is not divisible by p, this implies that $c-b$ is divisible by p.
Thus, we find that $b = a+(b-a) \geq a+p$ and $c = b+(c-b) \geq a+2p$.

We have $a \geq 2$. Indeed, suppose that $a = 1$. Then, both integers $b+1 = ab+1$ and $b-1 = b-a$ are divisible by p, which implies that their difference $(b+1)-(b-1) = 2$ is divisible by p as well. However, p is an odd prime and can therefore not divide 2.

Using $a \geq 2$, $b \geq a+p$, and $c \geq a+2p$, we conclude that

$$\frac{a+b+c}{3} \geq \frac{a+(a+p)+(a+2p)}{3} = p+a \geq p+2.$$

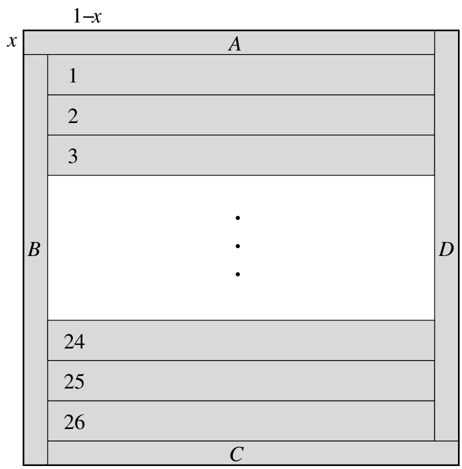
$\square$

b) Again, we may assume that $a < b < c$. In part a) we have seen that $\frac{a+b+c}{3} \geq \frac{a+(a+p)+(a+2p)}{3} = p + a \geq p + 2$. We can only have $\frac{a+b+c}{3} = p + 2$ if $b = a + p$, $c = a + 2p$, and $a = 2$. Since $ab + 1 = 2(2 + p) + 1 = 2p + 5$ must be divisible by p, it follows that 5 is divisible by p. We conclude that $p = 5$, $b = 7$, and $c = 12$. The quadruple $(p, a, b, c) = (5, 2, 7, 12)$ is indeed a Leiden quadruple, because $ab + 1 = 15$, $ac + 1 = 25$, and $bc + 1 = 85$ are all divisible by p.

We conclude that $p = 5$ is the only number for which there is a Leiden quadruple (p, a, b, c) that satisfies $\frac{a+b+c}{3} = p + 2$. □

5. a) Consider a rectangle with sides of length $a \leq b$ inside the square. Since $b \leq 1$ and $2a + 2b = \frac{5}{2}$ hold, we see that $a \geq \frac{1}{4}$. The area of the rectangle equals ab and is therefore at least $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$. Hence, we can have no more than 16 rectangles inside the square without creating overlaps. □

b) A solution is sketched in the figure below. The four outer rectangles, A through D, are equal with the shorter side having length x, and the longer side having length $1 - x$. Together they leave uncovered a square area with sides of length $1 - 2x$. This area is then tiled by 26 equal rectangles. These have sides of length $1 - 2x$ and $\frac{1-2x}{26}$, and therefore have a circumference of $\frac{54}{26}(1 - 2x)$. To obtain a circumference of length 2, we take $x = \frac{1}{54}$.



□

BxMO/EGMO Team Selection Test, March 2015

Problems

1. Let m and n be positive integers such that $5m + n$ is a divisor of $5n + m$. Prove that m is a divisor of n.

2. Given are positive integers r and k and an infinite sequence of positive integers $a_1 \leq a_2 \leq \ldots$ such that $\frac{r}{a_r} = k + 1$. Prove that there is a t satisfying $\frac{t}{a_t} = k$.

3. Let $n \geq 2$ be a positive integer. Each square of an $n \times n$-board is coloured red or blue. We put dominoes on the board, each covering two squares of the board. A domino is called *even* if it lies on two red or two blue squares and *colourful* if it lies on a red and a blue square. Find the largest positive integer k having the following property: regardless of how the red/blue-colouring of the board is done, it is always possible to put k non-overlapping dominoes on the board that are either all even or all colourful.

4. In a triangle ABC the point D is the intersection of the interior angle bisector of $\angle BAC$ and side BC. Let P be the second intersection point of the exterior angle bisector of $\angle BAC$ with the circumcircle of $\triangle ABC$. A circle through A and P intersects line segment BP internally in E and line segment CP internally in F. Prove that $\angle DEP = \angle DFP$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$(x^2 + y^2)f(xy) = f(x)f(y)f(x^2 + y^2)$$

for all real numbers x and y.

Solutions

1. There is a positive integer k with $(5m+n)k = 5n+m$. Hence, $5km-m = 5n-kn$, or $(5k-1)m = (5-k)n$. The left hand side is positive, hence also the right hand side is positive, which yields $k < 5$. If $k = 1$, then $4m = 4n$, hence $m = n$, hence $m \mid n$. If $k = 2$, then $9m = 3n$, hence $3m = n$, hence $m \mid n$. If $k = 3$, then $14m = 2n$, hence $7m = n$, hence $m \mid n$. If $k = 4$, then $19m = n$, hence $m \mid n$. We conclude that in all cases we have that $m \mid n$. □

2. We will prove this by contradiction. Suppose that such a t does not exist. If $a_k = 1$, then $\frac{k}{a_k} = k$ would hold, contradicting our assumption. Hence, $a_k \geq 2$. We will now prove by induction to i that $a_{ik} \geq i+1$. We just proved the base case. Now suppose that for certain $i \geq 1$ we have that $a_{ik} \geq i+1$. Then we also have that $a_{(i+1)k} \geq i+1$. If $a_{(i+1)k} = i+1$, then $\frac{(i+1)k}{a_{(i+1)k}} = k$, which is a contradiction. Hence, $a_{(i+1)k} \geq i+2$. This finishes the induction. Now take $i = a_r$, then we have $a_{a_r k} \geq a_r + 1$. Moreover, because $r = a_r(k+1)$ we have $a_r = a_{a_r(k+1)} \geq a_{a_r k} \geq a_r + 1$, which is a contradiction. □

3. We will prove that $k = \lfloor \frac{n^2}{4} \rfloor$ is the largest possible integer.

Suppose that n is even. Then it is possible to cover the board with $\frac{n^2}{2}$ dominoes (without considering the colours). Because there are $\frac{n^2}{2}$ dominoes, each of which is either colourful or even, there are at least $\lfloor \frac{n^2}{4} \rfloor = \frac{n^2}{4}$ colourful or at least $\lfloor \frac{n^2}{4} \rfloor = \frac{n^2}{4}$ even dominoes.
When n is odd, we can cover the board with $\frac{n^2-1}{2}$ dominoes. (Notice that this number is an even integer.) Of these dominoes either at least $\frac{n^2-1}{4} = \lfloor \frac{n^2}{4} \rfloor$ are colourful, or at least $\frac{n^2-1}{4} = \lfloor \frac{n^2}{4} \rfloor$ are even. This proves that it is always possible to put at least $\lfloor \frac{n^2}{4} \rfloor$ colourful or at least $\lfloor \frac{n^2}{4} \rfloor$ even dominoes on the board.

We will now prove that it is possible to create a colouring of the board with blue and red squares, such that no more than $\lfloor \frac{n^2}{4} \rfloor$ even and no more than $\lfloor \frac{n^2}{4} \rfloor$ colourful dominoes can be put on the board.

Colour the squares of the board in the colours white and black like the squares on a chess board, such that the lowerleft square is white. If n is even, there are equally many white as black square, namely $\frac{n^2}{2}$. If n is odd, there is one black square less and the number of black squares equals $\frac{n^2-1}{2} = \lfloor \frac{n^2}{2} \rfloor$. In both cases this is an even number of squares, as for odd n we have that $n^2 \equiv 1 \mod 4$. Now colour half of the black squares red and all the other squares blue. Then there are $\lfloor \frac{n^2}{4} \rfloor$ red squares, hence we can put at most $\lfloor \frac{n^2}{4} \rfloor$ non-overlapping colourful dominoes on the board as each of these dominoes covers one red square. An even domino cannot cover two red squares, because there are no pairs of adjacent squares coloured red. Hence, it must cover two blue squares. One of these blue squares must have been black, hence the number of even dominoes is at most the number of black-blue squares and that is $\lfloor \frac{n^2}{4} \rfloor$. Hence of both the colourful as the even dominoes we can put at most $\lfloor \frac{n^2}{4} \rfloor$ simultaneously on the board.

We conclude that the maximum k is indeed $k = \lfloor \frac{n^2}{4} \rfloor$. □

4. We consider the configuration in which the points A, C, B, and P lie in that order on the circumcircle. The other case is analogous.

By the inscribed angle theorem for the circumcircle of $\triangle ABC$, we have

$$\angle ABE = \angle ABP = \angle ACP = \angle ACF.$$

Moreover, by the inscribed angle theorem for the circle through A, P, E, and F:

$$\angle AEB = 180^\circ - \angle AEP = 180^\circ - \angle AFP = \angle AFC.$$

We therefore see (AA) that $\triangle ABE \sim \triangle ACF$. From this, it follows that

$$\frac{|AB|}{|AC|} = \frac{|BE|}{|CF|}.$$

The angle bisector theorem then implies that

$$\frac{|AB|}{|AC|} = \frac{|DB|}{|DC|}, \text{ so therefore } \frac{|BE|}{|CF|} = \frac{|DB|}{|DC|}. \tag{1}$$

Choose Z on PA such that A lies between P and Z. As AP is the external angle bisector of $\angle BAC$, we have $\angle PAB = \angle ZAC = 180^\circ - \angle PAC$. So using the inscribed angle theorem and the fact that $ACBP$ is a cyclic quadrilateral, we see that

$$\angle DCF = \angle PCB = \angle PAB = 180^\circ - \angle PAC = \angle PBC = \angle EBD.$$

If we combine this with (1), we obtain $\triangle BED \sim \triangle CFD$ (SAS). Therefore $\angle BED = \angle CFD$, hence we have $\angle DEP = 180^\circ - \angle BED = 180^\circ - \angle CFD = \angle DFP$. □

5. Substituting $x = y = 0$ gives $0 = f(0)^3$, so $f(0) = 0$. We consider two more cases; either f has another zero, or f has no other zeroes. In the first case there is some $a \neq 0$ such that $f(a) = 0$. Substituting $x = a$ then gives $(a^2 + y^2)f(ay) = 0$. As $a^2 + y^2 > 0$ (since $a \neq 0$) we find that $f(ay) = 0$ for all y. As ay can attain all values in $\mathbb{R}$, it follows that $f(x) = 0$ for all x; i.e. f is the constant function with value 0. Note that this function indeed satisfies the given equation.

So now consider the second case, in which $f(x) \neq 0$ for all $x \neq 0$. Substituting $x \neq 0$ and $y = 1$ then gives $(x^2 + 1)f(x) = f(x)f(1)f(x^2 + 1)$ and $f(x) \neq 0$, so we can divide this equation by $f(x)$. Therefore $(x^2 + 1) = f(1)f(x^2 + 1)$. Let $c = f(1)$. Note that $c \neq 0$, so $f(x^2 + 1) = \frac{x^2+1}{c}$. Since $x^2 + 1$ attains all reals that are at least 1, we find $f(x) = \frac{x}{c}$ for all $x > 1$.

Substituting $x = y = 2$ now gives $(4 + 4)f(4) = f(2)f(2)f(4 + 4)$. Since we know the values $f(x)$ takes for $x > 1$, we know the values of $f(2)$, $f(4)$, and $f(8)$. Therefore $8 \cdot \frac{4}{c} = \frac{2}{c} \cdot \frac{2}{c} \cdot \frac{8}{c}$, or equivalently, $\frac{1}{c} = \frac{1}{c^3}$. We deduce that $c^2 = 1$, so $c = 1$ or $c = -1$. As $c^2 = 1$, it follows that $f(x) = cx$ for all $x > 1$.

Let $x > 1$, and substitute $y = \frac{1}{x}$. This gives $(x^2 + \frac{1}{x^2})f(1) = f(x)f(\frac{1}{x})f(x^2 + \frac{1}{x^2})$. Since $x > 1$, we have $x^2 + \frac{1}{x^2} > 1$, so we deduce that $(x^2 + \frac{1}{x^2})c = xc \cdot f(\frac{1}{x}) \cdot c(x^2 + \frac{1}{x^2})$, or equivalently, $1 = xc \cdot f(\frac{1}{x})$. Therefore $f(\frac{1}{x}) = \frac{1}{xc} = c \cdot \frac{1}{x}$. We deduce that $f(x) = cx$ for all $x > 0$ with $x \neq 1$. Since we also have $f(1) = c$, it follows that $f(x) = cx$ for all $x > 0$.

Substituting $x = y = -1$ gives $2f(1) = f(-1)^2 f(2)$. Since $f(1) = c$ and $f(2) = 2c$, it follows that $2c = f(-1)^2 \cdot 2c$, so since $c \neq 0$ we have $f(-1)^2 = 1$. Therefore either $f(-1) = 1$ or $f(-1) = -1$.

Let $x > 0$. Substituting $y = -1$ then gives $(x^2+1)f(-x) = f(x)f(-1)f(x^2+1)$, so $(x^2 + 1)f(-x) = cx \cdot f(-1) \cdot c(x^2 + 1)$, or equivalently, $f(-x) = c^2 x f(-1) = xf(-1)$. Let $d = f(-1)$. Then $f(x) = -dx$ for all $x < 0$, where $d^2 = 1$.

So aside from $f(x) = 0$ there are four more possible solutions, namely $f(x) = x$, $f(x) = -x$, $f(x) = |x|$, and $f(x) = -|x|$. We first check $f(x) = tx$ with $t = \pm 1$. The left hand side then reads $(x^2 + y^2) \cdot txy$, and the right hand side then reads $tx \cdot ty \cdot t(x^2 + y^2)$. As $t^2 = 1$, the left hand side and the right hand side are equal, so these two functions are indeed solutions of the functional equation.

Next, we check the two functions $f(x) = t|x|$, with $t = \pm 1$. Now the left hand side reads $(x^2 + y^2) \cdot t|xy|$, and the right hand side reads $t|x| \cdot t|y| \cdot t|x^2 + y^2|$. Since $x^2 + y^2 = |x^2 + y^2|$, $|xy| = |x||y|$, and $t^2 = 1$, the left hand side and the right hand side are equal. So these two functions are indeed solutions of the functional equations. Therefore there are five solutions; $f(x) = 0$, $f(x) = x$, $f(x) = -x$, $f(x) = |x|$ and $f(x) = -|x|$. $\square$

IMO Team Selection Test 1, June 2015

Problems

1. In a quadrilateral $ABCD$ we have $\angle A = \angle C = 90°$. Let E be a point in the interior of $ABCD$. Let M be the midpoint of BE. Prove that $\angle ADB = \angle EDC$ if and only if $|MA| = |MC|$.

2. Find all polynomials $P(x)$ with real coefficients such that the polynomial

$$Q(x) = (x + 1)P(x - 1) - (x - 1)P(x)$$

is constant.

3. Let n be a positive integer. Consider sequences $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots, b_k$ such that $a_0 = b_0 = 1$ and $a_k = b_k = n$ and such that for all i such that $1 \leq i \leq k$, we have that (a_i, b_i) is either equal to $(1 + a_{i-1}, b_{i-1})$ or $(a_{i-1}, 1 + b_{i-1})$. Consider for $1 \leq i \leq k$ the number

$$c_i = \begin{cases} a_i & \text{if } a_i = a_{i-1}, \\ b_i & \text{if } b_i = b_{i-1}. \end{cases}$$

Show that $c_1 + c_2 + \cdots + c_n = n^2 - 1$.

4. Let Γ_1 and Γ_2 be circles – with respective centres O_1 and O_2 – that intersect each other in A and B. The line O_1A intersects Γ_2 in A and C and the line O_2A intersects Γ_1 in A and D. The line through B parallel to AD intersects Γ_1 in B and E. Suppose that O_1A is parallel to DE. Show that CD is perpendicular to O_2C.

5. For a positive integer n, we define D_n as the largest integer that is a divisor of $a^n + (a + 1)^n + (a + 2)^n$ for all positive integers a.

1. Show that for all positive integers n, the number D_n is of the form 3^k with $k \geq 0$ an integer.
2. Show that for all integers $k \geq 0$ there exists a positive integer n such that $D_n = 3^k$.

Solutions

1. Let N be the midpoint of BD. By Thales's Theorem the circle with diameter BD also passes through A and C, and N is the centre of this circle. Moreover, we have $MN \parallel DE$; if E doesn't lie on BD, then MN is a midparallel in triangle BDE, and if E does lie on BD, then MN and DE are equal.

The claim that $|AM| = |CM|$ is equivalent to that of M being on the perpendicular bisector of AC. As said bisector passes through N, the above claim is equivalent to $MN \perp AC$. Note that this is the case if and only if $DE \perp AC$. Let T be the intersection point of DE and AC; then $|AM| = |CM|$ if and only if $\angle DTC = 90°$.

Using the sum of angles of a triangle, we see that $\angle DTC = 180° - \angle TDC - \angle DCT = 180° - \angle EDC - \angle DCA$. We have $\angle DCA = 90° - \angle ACB = 90° - \angle ADB$, where we used in the last equality that $ABCD$ is a cyclic quadrilateral. Therefore $\angle DTC = 180° - \angle EDC - (90° - \angle ADB) = 90° - \angle EDC + \angle ADB$. Now it immediately follows that $\angle ADB = \angle EDC$ if and only if $\angle DTC = 90°$, which we already know to be equivalent to $|AM| = |CM|$.

Note that this proof does not depend on the configuration. $\square$

2. Suppose that $P(x)$ is a constant polynomial, say $P(x) = a$ with $a \in \mathbb{R}$. Then

$$Q(x) = (x + 1)a - (x - 1)a = ax + a - ax + a = 2a,$$

which is constant. So every constant $P(x)$ satisfies the condition.

Now we assume that P is not constant. We can then write $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 1$ and $a_n \neq 0$. Consider the x^n-coefficient in $Q(x)$. It is the sum of the x^n-coefficients of $xP(x-1)$, $P(x-1)$, $-xP(x)$, and $P(x)$. In the first of these, this coefficient equals $a_{n-1} - na_n$, in the second one it equals a_n, in the third one it equals $-a_{n-1}$ and the fourth one it equals a_n. Summing these gives

$$a_{n-1} - na_n + a_n - a_{n-1} + a_n = (2 - n)a_n.$$

But in $Q(x)$ this coefficient must be equal to 0. Since $a_n \neq 0$, it follows that $n = 2$. Therefore $P(x) = a_2 x^2 + a_1 x + a_0$ with $a_2 \neq 0$.

Now consider the constant coefficients of $Q(x)$. It is the sum of the constant coefficients of $xP(x - 1)$, $P(x - 1)$, $-xP(x)$, and $P(x)$. These are respectively 0, $a_2 - a_1 + a_0$, 0, and a_0. These sum up to $a_2 - a_1 + 2a_0$. On the other hand, we can compute $Q(1)$;

$$Q(1) = 2P(0) - 0 = 2a_0.$$

As $Q(x)$ is constant, it follows that $Q(1)$ is the constant coefficient of Q, so $2a_0 = a_2 - a_1 + 2a_0$. Therefore $a_2 = a_1$. So $P(x)$ is of the form $bx^2 + bx + a = bx(x+1) + a$ with $a, b \in \mathbb{R}$ and $b \neq 0$. To see whether these polynomials indeed satisfy the condition, we substitute this form for $P(x)$.

$$\begin{aligned} Q(x) &= (x+1)\big(b(x-1)x + a\big) - (x-1)(bx(x+1) + a) \\ &= (x-1)x(x+1)b + (x+1)a - (x-1)x(x+1)b - (x-1)a \\ &= 2a. \end{aligned}$$

This is indeed constant, so all such $P(x)$ satisfy the condition. In fact, every constant polynomial is also of this form, with $b = 0$. Therefore the polynomials satisfying the condition are precisely those of the form $P(x) = bx^2 + bx + a$ with $a, b \in \mathbb{R}$. $\square$

3. We prove by induction on j that $c_1 + \ldots + c_j = a_j b_j - 1$. For $j = 1$ this reads $c_1 = a_1 b_1 - 1$, which is true since $(a_1, b_1) \in \{(1,2),(2,1)\}$. Now suppose that $c_1 + \cdots + c_{i-1} = a_{i-1}bi - 1 - 1$. We assume without loss of generality that $(a_i, b_i) = (a_{i-1}, 1 + b_{i-1})$, so that $a_i = a_{i-1}$ and therefore $c_i = a_{i-1}$. We see that

$$(c_1 + \ldots + c_{i-1}) + c_i = (a_{i-1}b_{i-1} - 1) + a_{i-1} = a_{i-1}(b_{i-1} + 1) - 1 = a_i b_i - 1,$$

which completes the induction argument. Substituting $j = k$, it now follows that $c_1 + \cdots + c_k = a_k b_k - 1 = n^2 - 1$. $\square$

4. We consider only the configuration in which A, B, E, and D lie in that order on a circle, in which O_1, A, and C lie in that order on a line, and O_2, A, and D lie in that order on a line; the proof is analogous for the other configurations.

As $ABED$ is a cyclic quadrilateral, we have $\angle BED = 180° - \angle DAB$. Moreover, using the parallel lines, we see that $\angle BED = \angle DAO_1$ and as $|O_1A| = |O_1D|$, we have $\angle DAO_1 = \angle ADO_1$. We deduce that $180° - \angle DAB = \angle ADO_1$. Therefore DO_1 and AB are parallel.

We already know that $\angle ADO_1 = \angle DAO_1$. Since $|O_2A| = |O_2C|$, it follows that $\angle DAO_1 = \angle O_2AC = \angle O_2CA$. So $\angle O_2DO_1 = \angle ADO_1 = \angle O_2CA = \angle O_2CO_1$, so O_1DCO_2 is a cyclic quadrilateral.

The line O_1O_2 is the perpendicular bisector AB, therefore is also perpendicular to DO_1, as this line is parallel to AB. Therefore $\angle O_2O_1D = 90°$. As O_1DCO_2 is a cyclic quadrilateral, we now also have $\angle O_2CD = 90°$. □

5.

1. Let p be a prime and suppose that p divides D_n. Then p divides

$$\big((a+1)^n+(a+2)^n+(a+3)^n\big)-\big(a^n+(a+1)^n+(a+2)^n\big) = (a+3)^n-a^n$$

for all positive integers a.

Substituting $a = p$, then it follows that $p \mid (p+3)^n - p^n$, i.e. we have $(p+3)^n - p^n \equiv 0 \mod p$. This simply reads $3^n \equiv 0 \mod p$, so $p = 3$. We deduce that D_n only contains prime factors equal to 3, and therefore is of the form 3^k with $k \geq 0$ an integer.

2. For $k = 0$ we take $n = 2$. We have $1^2+2^2+3^2 = 14$ and $2^2+3^2+4^2 = 29$ and these two numbers are coprime, so $D_2 = 1$. Now assume that $k \geq 1$. We show that $D_n = 3^k$ for $n = 3^{k-1}$.

We first show that $1^n + 2^n + 3^n$ for $n = 3^{k-1}$ is divisible by 3^k, but not by 3^{k+1}. For $k = 1$ we have $n = 1$, and indeed, we see that $1+2+3 = 6$ is divisible by 3 but not by 3^2. For $k \geq 2$ we have $n > k$, so that 3^n is divisible by 3^{k+1}. So we are reduced to showing that $1 + 2^n$ for $n = 3^{k-1}$ is divisible by 3^k but not by 3^{k+1}. We show this by induction on k. For $k = 2$ we have $n = 3$, so indeed $1 + 8 = 9$ is divisible by 9, but not by 27. Let $m \geq 2$, and suppose we have proved our claim for $k = m$. Let $n = 3^{m-1}$. Then $1 + 2^n$ is divisible by 3^m, but not by 3^{m+1}. It suffices to show that $1+2^{3m}$ is divisible by 3^{m+1}, but not by 3^{m+2}. Write $1 + 2^n = 3^m c$ with $3 \nmid c$. Then $2^n = 3^m c - 1$, so

$$1 + 2^{3n} = 1 + (3^m c - 1)^3 = 3^{3m}c^3 - 3 \cdot 3^{2m}c^2 + 3 \cdot 3^m c.$$

Modulo 3^{m+2}, this is congruent to $3^{m+1}c$, and since $3 \nmid c$, it follows that this is divisible by 3^{m+1}, but not by 3^{m+2}, as desired. This completes our inductive argument.

Next we show that for $n = 3^{k-1}$, we have that $(a+3)^n - a^n$ is divisible by 3^k for all positive integers a. Again, we prove this by induction on k. For $k = 1$ we have $n = 1$, so indeed we see that $(a + 3) - a = 3$ is divisible by 3 for all positive integers a. Now suppose that $m \geq 1$, and suppose that we proved our claim for $k = m$. Let $n = 3^{m-1}$. Then $(a + 3)^n - a^n$ is divisible by 3^m for all positive integers a, so we can write $(a + 3)^n = a^n + 3^m c$ for some integer c. Taking third powers of both sides then yields

$$(a + 3)^{3n} = a^{3n} + 3a^{2n} \cdot 3^m c + 3a^n \cdot 3^{2m} c^2 + 3^{3m} c^3,$$

so

$$(a + 3)^{3n} - a^{3n} = a^{2n} \cdot 3^{m+1} c + a^n \cdot 3^{2m+1} c^2 + 3^{3m} c^3,$$

which is divisible by 3^{m+1}. This completes our inductive argument.

We have now shown for $n = 3^{k-1}$ that $3^k \mid 1^n + 2^n + 3^n$ and $3^k \mid (a + 3)^n - a^n$ for all positive integers a, from which we, by induction on a, immediately deduce that $3^k \mid a^n + (a + 1)^n + (a + 2)^n$ for all positive integers a. Therefore $3^k \mid D_n$. As $3^{k+1} \nmid 1^n + 2^n + 3^n$, we also have $3^{k+1} \nmid D_n$. Therefore $D_n = 3^k$, as desired. $\square$

IMO Team Selection Test 2, June 2015

Problems

1. Let a and b be two positive integers satifying $\gcd(a, b) = 1$. Consider a pawn standing on the grid point (x, y). A step of type A consists of moving the pawn to one of the following grid points: $(x+a, y+a), (x+a, y-a), (x-a, y+a)$ or $(x-a, y-a)$. A step of type B consists of moving the pawn to $(x+b, y+b), (x+b, y-b), (x-b, y+b)$ or $(x-b, y-b)$.
Now put a pawn on $(0, 0)$. You can make a (finite) number of steps, alternatingly of type A and type B, starting with a step of type A. You can make an even or odd number of steps, i.e., the last step could be of either type A or type B. Determine the set of all grid points (x, y) that you can reach with such a series of steps.

2. Determine all positive integers n for which there exist positive integers a_1, a_2, ..., a_n with

$$a_1 + 2a_2 + 3a_3 + \ldots + na_n = 6n$$

and

$$\frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + \ldots + \frac{n}{a_n} = 2 + \frac{1}{n}.$$

3. An equilateral triangle ABC is given. On the line through B parallel to AC there is a point D, such that D and C are on the same side of the line AB. The perpendicular bisector of CD intersects the line AB in E. Prove that triangle CDE is equilateral.

4. Each of the numbers 1 up to and including 2014 has to be coloured; half of them have to be coloured red the other half blue. Then you consider the number k of positive integer that are expressible as the sum of a red and a blue number. Determine the maximum value of k that can be obtained.

5. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $f(1) = 2$ and such that for all $m, n \in \mathbb{Z}_{>0}$ we have that $\min(2m + 2n, f(m + n) + 1)$ is divisible by $\max(f(m) + f(n), m + n)$.

Solutions

1. We will prove that the grid point (x, y) is reachable if and only if $x+y \equiv 0 \mod 2$.

If we move the pawn from (x, y) to $(x \pm a, y \pm a)$, then the sum of the new coordinates equals $x+y+2a$, $x+y$ or $x+y-2a$, hence it is congruent to the sum of the old coordinates modulo 2. The same holds when making a step of type B. Because the pawn starts at $(0, 0)$, after executing any finite number of steps, we will have that the sum of the coordinates of the grid point where the pawn is standing, is even. Hence, the points (x, y) with $x+y \equiv 1 \mod 2$ are not reachable.

Now we will show that all other points are reachable. Because $\gcd(a, b) = 1$, there exist integers m, n with $ma+nb = 1$. Then we have $2ma+2nb = 2$. Of the numbers m and n one must be positive and the other negative. We assume that m is positive. The other case is treated analogously. We will now first make $2m$ steps of type A and $2m$ steps of type B. For the steps of type A we choose to do m times the step $(x, y) \mapsto (x+a, y+a)$ and m times $(x, y) \mapsto (x+a, y-a)$. For the steps of type B we choose m times $(x, y) \mapsto (x+b, y+b)$ and m times $(x, y) \mapsto (x-b, y-b)$. The effect of all these steps together is that the x-coordinate increased by $2ma$ and the y-coordinate did not change (in face, the B-steps cancel each other). After this we do $2|n|$ steps of type A and $2|n|$ steps of type B. For the steps of type A we choose to do $|n|$ times the step $(x, y) \mapsto (x+a, y+a)$ and $|n|$ times $(x, y) \mapsto (x-a, y-a)$. For the steps of type B we choose to do $|n|$ times $(x, y) \mapsto (x-b, y+b)$ and $|n|$ times $(x, y) \mapsto (x-b, y-b)$. The effect of all these steps together is that the x-coordinate decreased by $2|n|$ and the y-coordinate did not change (in fact, now the A-steps cancel each other). Altogether, after these $4m+4|n|$ steps, we moved the pawn from a starting point (x, y) to the point $(x+2ma-2|n|b, y) = (x+2, y)$. We can construct an analogous series of steps that move the pawn from (x, y) to $(x, y+2)$, and also analogous series of steps that move the pawn from (x, y) to $(x-2, y)$ and $(x, y-2)$. Each of these series starts with a step of type A and ends with a step of type B.

By combining series like these, we can move the pawn from $(0, 0)$ to any point (x, y) with $x \equiv y \equiv 0 \mod 2$. Now consider a point (x, y) with $x \equiv y \equiv 1 \mod 2$. Because $\gcd(a, b) = 1$, at least one of a and b is odd. Suppose that a is odd. Then the point $(x-a, y-a)$ is a point with two even coordinates, to which we can construct a series of steps, ending with a step of type B. After that, we can make a step of type A that moves the pawn from $(x-a, y-a)$ to (x, y). Next, suppose that a is even. Then b is odd. The point $(x-b, y-b)$ now has two even coordinates, hence we can reach this point with a series of steps ending with a step of type B. Now we execute three more steps after this:

$$(x-b, y-b) \overset{\text{A}}{\longmapsto} (x-b-a, y-b-a) \overset{\text{B}}{\longmapsto} (x-a, y-a) \overset{\text{A}}{\longmapsto} (x, y).$$

In this way we can also reach the point (x, y).

We conclude that we can reach all points (x, y) with $x+y \equiv 0 \mod 2$ and we cannot reach any other point. □

2. If we apply the inequality of the arithmetic and harmonic mean to a_1, two copies of a_2, three copies of a_3, $\ldots$, n copies of a_n, then we find that

$$\frac{6n}{\frac{1}{2}n(n+1)} = \frac{a_1 + 2a_2 + \ldots + na_n}{\frac{1}{2}n(n+1)} \geq \frac{\frac{1}{2}n(n+1)}{\frac{1}{a_1} + \frac{2}{a_2} + \ldots + \frac{n}{a_n}} = \frac{\frac{1}{2}n(n+1)}{2 + \frac{1}{n}}.$$

We have

$$\frac{6n}{\frac{1}{2}n(n+1)} = \frac{12}{n+1} < \frac{12}{n}$$

and

$$\frac{\frac{1}{2}n(n+1)}{2+\frac{1}{n}} = \frac{\frac{1}{2}n^2(n+1)}{2n+1} > \frac{\frac{1}{2}n^2(n+1)}{2n+2} = \tfrac{1}{4}n^2.$$

Altogether we find that $\frac{12}{n} > \frac{1}{4}n^2$, or $48 > n^3$, which yields $n \leq 3$.

For $n = 1$ we get $a_1 = 6$ and $\frac{1}{a_1} = 3$, which is a contradiction. Hence, $n = 1$ is not possible.

For $n = 2$ we get $a_1 + 2a_2 = 12$ and $\frac{1}{a_1} + \frac{2}{a_2} = 2 + \frac{1}{2}$. If $a_2 \geq 2$ we have $\frac{1}{a_1} + \frac{2}{a_2} \leq 1 + 1$ and this is too small. Hence, we have $a_2 = 1$, but then we find that $a_1 = 12 - 2 = 10$ and hence $\frac{1}{a_1} + \frac{2}{a_2} = \frac{1}{10} + 2$, which is a contradiction. Hence, $n = 2$ is impossible.

For $n = 3$ there is a solution, namely $a_1 = 6$, $a_2 = 3$ and $a_3 = 2$. Hence, $n = 3$ is possible and we conlude that $n = 3$ is the only solution. □

3. We consider the configuration in which E lies between A and B. The case in which B lies between A and E is treated analogously. (Because of the condition that D and C lie on the same side of AB, it is impossible that A lies between B and E, hence we have treated all cases.)

As E lies on the perpendicular bisector of CD, we have $|EC| = |ED|$. Hence, it is sufficient to prove that $\angle CED = 60°$. First suppose that $E = B$. Then we have $\angle CED = \angle CBD = \angle ACB = 60°$ because of alternating (Z) angles, hence we are done. Now suppose that $E \neq B$.

As BD is parallel to AC, we have $\angle CBD = \angle ACB = 60° = \angle CBA$. Hence, the point E is the intersection point of the perpendicular bisector of CD and the exterior angle bisector of $\angle CBD$. This means that E lies on the circumcircle of triangle CDB. *(This is a known fact, but it is also possible to prove it as follows. Let E' be the intersection point of the exterior angle bisector of $\angle CBD$ and the circumcircle of $\triangle CBD$. Because BE' is the exterior angle bisector, we have $\angle CBE' = 180° - \angle DBE'$. Hence, chords CE' and DE' have the same length, which means that E' lies on the perpendicular bisector of CD.)* We conclude that $CEBD$ is a cyclic quadrilateral. Hence, $\angle CED = \angle CBD = 60°$. $\square$

4. Let $n = 2014$. We shall prove that the maximum k equals $2n - 5$. The smallest number that you could possibly write as the sum of a red and a blue number is $1 + 2 = 3$ and the largest number is $(n - 1) + n = 2n - 1$. Hence, there are at most $2n - 3$ numbers expressible as the sum of a red and a blue number.

Suppose that the numbers can be coloured in such a way that $2n - 3$ or $2n - 4$ of numbers are expressible as the sum of a red and a blue number. Now at most one of the numbers from 4 up to and including $2n - 1$ is not expressible in such a way. We will now show that we may assume without loss of generallity that this number is at least $n + 1$. Indeed, we could make a second colouring in which a number i is blue if and only if $n+1-i$ is blue in the initial colouring. Then in the case of the second colouring a number m is expressible as the sum of a red and a blue number if and only if $2n + 2 - m$ was expressible as the sum of a red and a blue number in the initial colouring. Hence, if in the initial colouring a number smaller than $n+1$ is not expressible as the sum of red and blue, then in the second colouring a number greater than $2n+2-(n+1) = n+1$ is not expressible as the sum of red and blue.

Hence, we may assume that the numbers 3 up to and including n are all expressible as the sum of red and blue. Because red and blue are interchangable, we may also assume without loss of generality that 1 is coloured blue. Because 3 is expressible as the sum of red and blue and this can only be $3 = 1 + 2$, the number 2 must be red. Now suppose that we know that 2 up to and including l are red, for certain l with $2 \leq l \leq n-2$. Then in all the possible sums $a + b = l + 2$ with $a, b \geq 2$ both numbers are colored red. However, we know that we can express $l + 2$ as the sum of red and blue (because $l + 2 \leq n$), hence that must be $1 + (l + 1)$. Hence, $l + 1$ is also coloured red. By induction, we now see that the numbers 2 up to and including $n - 1$ are all red. These are $n - 2 = 2012$ numbers. However, only $\frac{1}{2}n = 1007$ numbers are red, which is a contradiction.

We conclude that at least two numbers of 3 up to and including $2n-1$ are not expressible as the sum of a red and a blue number. We shall now show that we can colour the numbers in such a way that all number from 4 up to and inlcuding $2n-2$ are expressible as the sum of a red and a blue number, implying that the maximum k equals $2n-5$.

To obtain this, colour all the even numbers, except n, and also the number 1 blue. All odd numbers, except 1, and also the number n we colour red. By adding 1 to an odd number (unequal to 1) we can obtain all even numbers from 4 up to and inlcuding n as the sum of a red and a blue number. By adding 2 to an odd number (unequal to 1), we can obtain all odd numbers from 5 up to and including $n+1$ as the sum of a red and a blue number. By adding $n-1$ to an even number (unequal to n), we can obtain all odd numbers from $n+1$ up to and including $2n-3$ as the sum of a red and a blue number. By adding n to an even number (unequal to n), we can obtain all even numbers from $n+2$ up to and including $2n-2$ as the sum of a red and a blue number. Altogether, we can express all numbers from 4 up to and including $2n-2$ as the sum of a red and a blue number.

We conclude that the maximum k equals $2n-5=4023$. $\square$

5. By substituting $m=n$ we get that $\min(4n, f(2n)+1)$ is divisible by $\max(2f(n), 2n)$. That is, a number that is at most $4n$ is divisible by another number, which is at least $2f(n)$. This yields that $4n \geq 2f(n)$, hence $f(n) \leq 2n$ for all n.

By substituting $m=n=1$ we get that $\min(4, f(2)+1)$ is divisible by $\max(2f(1), 2) = \max(4, 2) = 4$. Hence, $\min(4, f(2)+1)$ cannot by smaller than 4, hence $f(2)+1 \geq 4$. But we have already deduced that $f(2) \leq 2 \cdot 2 = 4$, hence either $f(2) = 3$ or $f(2) = 4$ holds.

First suppose that $f(2) = 3$. We will prove by induction to n that $f(n) = n+1$ for all n. Namely, suppose that this holds for $n = r-1$ for some $r \geq 3$ and subtitute $m = 1$ and $n = r-1$. Then we get that $\min(2r, f(r)+1)$ is divisible by $\max(f(1)+f(r-1), r) = \max(r+2, r) = r+2$. As $\min(2r, f(r)+1) \leq 2r < 2(r+2)$, it holds that $\min(2r, f(r)+1) = r+2$. Because $r \geq 3$, we have $2r > r+2$, hence we have $f(r)+1 = r+2$, or $f(r) = r+1$. This finishes the induction. Hence, we find the candidate function $f(n)+1$. We will check immediately whether this function satisfies the conditions. For all $m, n \in \mathbb{N}$ we have that $\min(2m+2n, f(m+n)+1) = \min(2m+2n, m+n+2) = m+n+2$ as $m, n \geq 1$, and $\max(f(m)+f(n), m+n) = \max(m+n+2, m+n) = m+n+2$. The former is divisible by the latter (as it equals the latter), hence this function satisfies.

Now suppose that $f(2) = 4$. We will prove by induction to n that $f(n) = 2n$ for all n. Namely, suppose that this holds for $n = r - 1$ for some $r \geq 4$. We will prove that also $f(r) = 2r$ holds. First substitute $m = 1$ and $n = r - 1$. Then we find that $\min(2r, f(r) + 1)$ is divisible by $\max(f(1) + f(r-1), 1 + r - 1) = \max(2r, r) = 2r$. Hence, $\min(2r, f(r)+1)$ cannot be smaller than $2r$, hence $f(r) \geq 2r - 1$. However, we already knew that $f(r) \leq 2r$, hence $f(r) \in \{2r-1, 2r\}$. Suppose that $f(r) = 2r-1$. Then substitute $m = 1$ and $n = r$. Then we find that $\min(2(r+1), f(r+1)+1)$ is divisible by $\max(f(1) + f(r), 1 + r) = \max(2 + 2r - 1, r + 1) = 2r + 1$. Because $2r + 1 \nmid 2(r + 1)$, the minimum does not equal $2(r + 1)$, hence we have $f(r+1)+1 < 2r+2$ and, moreover, it must also be divisible by $2r+1$, hence $f(r + 1) = 2r$. Now also substitute $m = 2$ and $n = r - 1$. Then we find that $\min(2(r+1), f(r+1)+1) = \min(2r+2, 2r+1) = 2r+1$ is divisible by $\max(f(2) + f(r-1), 1+r) = \max(4+2r-2, r+1) = 2r+2$, which is a contradiction,. We conclude that $f(r) = 2r$, which finishes the induction. Hence, we find the candidate function $f(n) = 2n$ for all n. We will check whether this function satisfies the conditions. In this case, for all m, n we have that $\min(2m+2n, f(m+n)+1) = \min(2m+2n, 2m+2n+1) = 2m+2n$ and $\max(f(m) + f(n), m + n) = \max(2m + 2n, m + n) = 2m + 2n$. The former is divisible by the latter (because it equals the latter), hence this function satisfies the conditions.

We conclude that there are exactly two solutions: the function given by $f(n) = n + 1$ for all n and the function given by $f(n) = 2n$ for all n. $\square$

Junior Mathematical Olympiad, October 2014

Problems

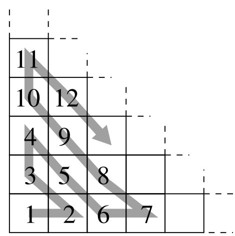
Part 1

1. Note that $555555 : 7 = 79365$. Consider the number $55 \cdots 55$ consisting of 1000 fives.
What is the remainder of this number on division by 7?

A) 2 B) 3 C) 4 D) 5 E) 6

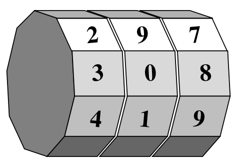
2. A pawn is placed on a board consisting of ten squares, numbered from 1 up to 10. The pawn is allowed to move from the square it is on to a square that either has a number that is two less, or a number that is twice as large. The pawn wants to make a sequence of moves that visits as many squares as possible. It may freely choose its starting point. It may visit squares multiple times. How many squares can the pawn visit in a single sequence of moves?

A) 6 B) 7 C) 8 D) 9 E) 10



3. Jan has huge square table of which the cells are numbered as in the figure. Which of the following five numbers does not occur in the leftmost column?

A) 55 B) 105 C) 172 D) 212 E) 300

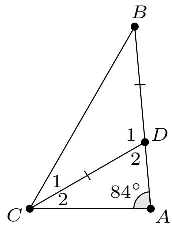


4. Birgit has a combination lock that consists of three rings next to one another, each having the digits 0 up to 9 in order. She turns the three rings until her secret combination is visible. Aside from this combination, there are 9 more combinations visible on the three rings. Coincidentally, one of these numbers is three times the secret combination.
What is Birgit's secret combination?

A) 106 B) 123 C) 272 D) 318 E) 328

5. In a triangle ABC, we have $\angle A = 84°$. Moreover, D is a point on the line segment AB such that $\angle D_1 = 3 \cdot \angle C_2$ and such that the line segments DC and DB have equal lengths.
What is $\angle C_1$?

A) 27° B) 28° C) 30° D) 32° E) 36°



6. A piece of apple pie had been stolen, and five children are being questioned on this. They all know who the culprit is, but not all of them are speaking the truth. Whenever one of the children lies, the next one will feel so guilty about this that he or she will tell the truth. The children make the following claims in the order shown:

- Asim: "Coen and I both didn't do it."
- Bob: "Either Coen or Dilan is the culprit."
- Coen: "Eva and I both didn't do it."
- Dilan: "Asim is the culprit."
- Eva: "At least two of Asim, Bob, Coen, and Dilan lied."

Who stole the apple pie?

A) Asim B) Bob C) Coen D) Dilan E) Eva

7. Consider the numbers $a = (3^4)^5$, $b = (4^4)^4$, and $c = (5^4)^3$. If you sort a, b and c from smallest to largest, you obtain:

A) $a < b < c$ B) $a < c < b$ C) $b < a < c$
D) $c < a < b$ E) $c < b < a$

8. Max has a lot of white and red paint. He starts with a 2-litre bucket in which one litre of red paint and one litre of white paint. Max now repeats the following step a number of times.

Step. Max pours precisely one litre out of the bucket, into a large container. Next, he fills the bucket back up to 2 litres of paint, using either the white paint, or the red paint. After this, he mixes the paint in the bucket.

After a number of steps, the percentage of red paint in the bucket must be between 83 and 84 percent. What is the smallest number of steps Max needs to attain this?

A) 5 B) 6 C) 7
D) 8 E) Max cannot obtain such a percentage.

Part 2

1. A member of a group of ten friends buys a bag of candy to share among the group. First he himself, who likes candy more than the rest of the group, takes a quarter of the candy. Another member grabs 30 pieces of candy. A third member grabs 10% of what is left. The remainder of the group distributes the remainder of the candy evenly. The total number of pieces of candy was less than 500 and everyone got at least one piece of candy.
How many pieces of candy were there in the bag?

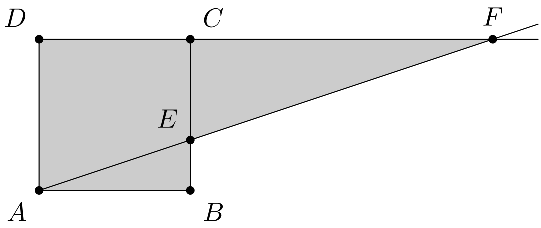
2. There are 36 balls, numbered from 1 up to 36. We want to put these into boxes in such a way that the following two conditions are satisfied:

(1) Every box contains at least 2 balls.

(2) Whenever you pick up two balls from a box, the sum of the two numbers of these balls is always a multiple of 3.

What is the smallest number of boxes for which this is possible?

3. We are given a square $ABCD$. A line is drawn through A that intersects the segment BC in E, and the line through C and D in F. The ratio of the lengths of the segments BE and EC is $1 : 2$. The area of the grey area is 60.
What is the area of the square?

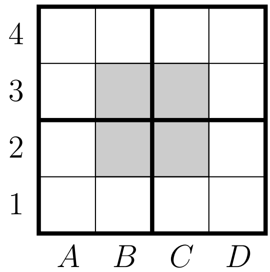


4. We want to exchange a 200-euro bill for bills of 5, 10, and 20 euros. One possibility is to exchange it for 5 bills of 20 euros, 6 bills of 10 euros, and 8 bills of 5 euros. Another possibility is to exchange it for 20 bills of 10 euros.
How many possibilities are there to exchange a 200-euro bill for bills of 5, 10, and 20 euros?

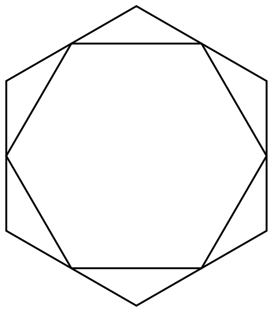
5. By stacking small cubes (all of the same size) neatly, a larger cube is formed. Two small cubes with faces placed against one another are called *neighbours*. So a cube can have at most six neighbours. The number of cubes having precisely four neighbours is 96.
How many small cubes are there having precisely five neighbours?

6. Michelle colours each of the numbers from 1 up to 2014. The first two numbers (1 and 2) are coloured red, the next two (3 and 4) are coloured white, the following two (5 and 6) are coloured blue, the two following those are coloured red, the two after those are coloured white, the two after that are coloured blue, and so on and so forth. Michelle then sums all blue numbers, and subtracts from that the sum of all red numbers.
What is the result?

7. The figure below represents a puzzle. The goal is to fill each of the 16 cells with a number from 1 up to 4. This has to be done in such a way that in each column and in each row, the four numbers are distinct. Moreover, in each of the four 2×2-squares, the four numbers have to be distinct as well. Finally, the four numbers in the grey squares also need to be distinct.
How many solutions does this puzzle have?



8. Mies has drawn a regular hexagon with area 1. She notices that the midpoints of the six sides also form a regular hexagon.
What is the area of this small hexagon?



Solutions

Part 1

1.	C) 4	**5.**	A) $27°$
2.	D) 9	**6.**	B) Bob
3.	D) 212	**7.**	D) $c < a < b$
4.	E) 328	**8.**	B) 6

Part 2

1.	320	**5.**	384
2.	13	**6.**	-1343
3.	36	**7.**	168
4.	121	**8.**	$\frac{3}{4}$

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