$-(x)=\frac{-1}{x^{2}}-x$
$x^{-}(x)=2 x^{3}$

$$
\begin{aligned}
& 0(x)=2 x^{3}+r(x)=-6 \\
& C(s)-6 x^{-9} r(1)=20
\end{aligned}
$$

$$
\begin{equation*}
2^{1} \quad 31 \tag{41}
\end{equation*}
$$

$$
\begin{array}{r}
4(s)-6 x^{-9}+8(1)=20 \\
4 x=1
\end{array}
$$

18

# for breakfast. 

Preferably unsolved ones...
$61^{\text {st }}$ Dutch Mathematical Olympiad 2022

## NEDERLANDSE WISKUNDE OLYMPIADE

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## Introduction

The selection process for IMO 2023 in Japan started with a first round in January 2022, held locally at all participating schools. This first round consists of eight multiple choice questions and four open questions, to be solved within 2 hours. This year 4388 students from 267 secondary schools participated in the first round.

The 782 best students were invited to the second round, which was held at twelve universities throughout the country in March 2022. This round consists of five open questions, and two problems for which the students have to give extensive solutions and proofs. The contest lasts 2.5 hours.

The 120 best students were invited to the final round, together with some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad. In total, 146 students were invited. In the preceding months, we organize four training sessions at each of the universities to help them prepare for their participation in the final round.

The final, in September, contains five problems for which the students has to give extensive solutions and proofs. They are allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 61st edition 2022.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad were invited to an intensive seven-month training programme. The students met twice for a three-day training camp, three times for a single day, and finally for a six-day training camp in the beginning of June. They also worked on weekly problem sets to be sent in to a personal trainer by email.

In February a team of four girls was chosen from the training group to represent the Netherlands in April at the EGMO in Slovenia. At this event the Dutch team won four bronze medals. For more information about the EGMO (including the 2023 paper), see www.egmo.org.

In March a selection test of 3.5 hours was held to determine the ten students from the training program which are sent to the Benelux Mathematical Olympiad (BxMO) held in May. One of the Dutch team members, Lance Bakker, managed to receive a full score here. For more information about the BxMO (including the 2023 paper), see www.bxmo.org.

Begin June the team for the International Mathematical Olympiad was selected by three team selection tests on three consecutive days, each lasting 4 hours. In addition to the six team members a seventh, young, promising student was selected to accompany the team to the IMO as an observer C. Two weeks later, the team had a training camp in Leiden and Tokyo from

June 27 to July 6.
For younger students the Junior Mathematical Olympiad was held in September 2022 at the VU University Amsterdam. The students invited to participate in this event were the 100 best students of grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with eight multiple choice questions and one with eight open questions.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

## Dutch delegation

The Dutch team for 64th edition of IMO 2023 in Japan consists of

- Lance Bakker (16 years old)
- silver medal at BxMO 2022, gold medal at BxMO 2023
- honourable mention at IMO 2022
- Bas Capel (17 years old)
- bronze medal at BxMO 2023
- Daan de Groot (18 years old)
- silver medal at BxMO 2022, bronze medal at BxMO 2023
- Hylke Hoogeveen (17 years old)
- bronze medal at BxMO 2020, honourable mention at BxMO 2021
- bronze medal at IMO 2021
- Mads Kok (16 years old)
- silver medal at BxMO 2022 and 2023
- honourable mention at IMO 2022
- Yanniek Nitescu (17 years old)
- bronze medal at BxMO 2022 and 2023

Also part of the IMO delegation, but not officially part of the IMO team, is:

- Allie Zong (17 years old)
- bronze medal at EGMO 2021 and 2023; silver medal at EGMO 2022

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Johan Konter (deputy leader), Leiden University
- Nils Van de Berg (observer B), Utrecht University


## First Round, January 2022

## Problems

## A-problems

1. A group of islands consists of a large, a medium and a small island. The total area of the three islands together is $23 \mathrm{~km}^{2}$. The difference between the areas of the large island and the medium island turns out to be exactly $1 \mathrm{~km}^{2}$ more than the area of the small island.
How many $\mathrm{km}^{2}$ is the area of the large island?
A) 10
B) 11
C) 12
D) 13
E) 14
2. Kevin draws a point $P$ on a large piece of paper. Then he draws, one by one, straight lines through $P$. How many lines does Kevin have to draw at least to make sure that on the piece of paper there are two
 lines that make an angle of less than 13 degrees?
A) 9
B) 13
C) 14
D) 27
E) 28
3. Sofie and her grandmother both have their birthday on 1 January. In six consecutive years, the age of grandma is an integer multiple of the age of her granddaughter Sofie. In the seventh year this is not the case. A few years later the age of grandma is again an integer multiple of the age of Sofie.

How old can grandma be by then?
A) 63
B) 66
C) 70
D) 90
E) 91
4. When you add the digits of the number 2022, you get 6 . How many 4-digit numbers are there (including 2022) such that, when you add the digits, you get 6 ? The numbers are not allowed to start with the digit 0 .
A) 40
B) 45
C) 50
D) 55
E) 56
5. Consider the equilateral triangle $P Q R$. Inside this triangle the regular hexagon $A B C D E F$ is drawn. Points $B, D$ and $F$ are the midpoints of the sides of the triangle $P Q R$. The area of the pentagon $Q B A F R$ is equal to 1 .
What is the area of the triangle $P Q R$ ?
A) $\frac{11}{10}$
B) $\frac{7}{6}$
C) $\frac{6}{5}$
D) $\frac{5}{4}$
E) $\frac{4}{3}$

6. A box contains red, white and blue balls. The number of red balls is an even number and the total number of balls in the box is less than 100. The number of white and blue balls together is exactly 4 times the number of red balls. The number of red and blue balls together is exactly 6 times the number of white balls.
How many balls are in the box?
A) 28
B) 30
C) 35
D) 70
E) 84
7. In a tournament with the four teams $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D , every team played against every other team in three rounds of two simultaneous games. No team won or lost all their games and no game ended in a draw. It is known that team A won in the first and third round. Also, team C won in the first round and team D lost in the second round. Five people make a statement about the tournament, but only one of them is telling the truth.
Which statement is true?
A) A and B played against each other in round 1
B) $C$ won against $B$
C) A and D played against each other in round 3
D) D won against A
E) B and C played against each other in round 2
8. Michael prints the net in the figure twice on cardboard and makes it into two identical dice, such that the pips are visible on the outside of the dice. He puts one dice on top of the other to make a small tower. The front face of the lower dice shows 3 pips. The total number of pips on the two faces touching in the middle is equal to 9 . The total number of pips on the back of the small tower is three times the total number of pips on the right side of the small tower.


How many pips are on the face that touches the ground?
A) 1
B) 2
C) 4
D) 5
E) 6

## B-problems

The answer to each B-problem is a number.

1. Line up the numbers 1 to 15 such that if you add any two numbers that are next to each other, you get a square number.
What do you get if you add the first and last number from the line?
2. In the figure below the large square has sides of length 6 . The circle is tangent to all sides of the large square. The four triangles are exactly the same right angled triangles and are directly next to each other; the small square they enclose has its vertices exactly on the circle.


What is the area of the grey triangle?
3. At a congress all attendees are either a mathematician or a biologist and there is no one that is both. The mathematicians all know each other and each of them knows four of the biologists. The biologists also all know each other and each of them knows nine of the mathematicians. It turns out that every mathematician knows twice as many people as every biologist. (If person A knows person B, then person B also knows person A.)
How many mathematicians are at the congress?
4. On an $8 \times 8$-board there is a beetle on every square. At a certain moment the distribution of the beetles on the board changes: every beetle crawls either one square to the left or one square diagonally to the bottom right. If a beetle can make neither of the two movements without falling off the board, it stays on its square.
At most how many squares can end up empty by this change?

## Solutions

A-problems

1. C) 12
2. C) $\frac{6}{5}$
3. C) 14
4. D) 70
5. C) 70
6. B) C won against B
7. E) 56
8. A) 1
B-problems
9. 17
10. $4 \frac{1}{2}$
11. 117
12. 29

## Second Round, March 2022

## Problems

## B-problems

The answer to each B-problem is a number.

B1. In a $3 \times 2$ rectangle, we put the numbers 1 to 6 in the squares in such a way that each number occurs exactly once. The score of such a distribution is determined as follows: for each two adjacent squares we compute the difference between their two numbers
 and we add up all these differences. In the example on the right, the differences are indicated in red. This distribution has score 17.
What is the smallest possible score of such a distribution?

B2. For how many integers $n$ with $1 \leq n \leq 800$ is the number $8 n+1$ a square?

B3. We start with a square with side length 1. During the first minute, small squares with side length $\frac{1}{3}$ grow on the middle of the vertical sides. During the next minute, on the middle of each vertical line segment in the new figure, a new small square grows, whose sides have length $\frac{1}{3}$ of these line segments. Below you can see the situation after 0,1 , and 2 minutes.


This process continues like this. Each minute, on the middle of each vertical line segment a new square grows, whose sides are $\frac{1}{3}$ of the length of that line segment. After one hour this process of new squares growing on the figure has happened 60 times.
What is the circumference of the figure after one hour?

B4. The candy store sells chocolates in the flavours white, milk, and dark. You can buy them in three types of coloured boxes. The three boxes have the following contents:

- Gold: 2 white, 3 milk, 1 dark,
- Silver: 1 white, 2 milk, 4 dark,
- Bronze: 5 white, 1 milk, 2 dark.

Lavinia buys some boxes of chocolates (at least one) and when she gets home, it turns out she has exactly the same number of chocolates of each flavour.

At least how many boxes did Lavinia buy?

B5. In triangle $A B C$, angle $A$ is a right angle. A point $D$ lies on line segment $A B$ in such a way that the angles $A C D$ and $B C D$ are equal. Moreover, $|A D|=2$ and $|B D|=3$.
What is the length of line segment $C D$ ?


C-problems For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. Alicia writes down $a$ distinct integers on a piece of paper and Britt writes down $b$ distinct integers on another piece of paper. Alicia wrote down at least one integer that Britt did not write down, and Britt wrote at least one integer down that Alicia did not write down. Vera counts the number of distinct integers on the two pieces of paper; let this number of distinct integers be $v$. Daan counts how many of the integers that have been written down by Alicia, have also been written down by Britt; let $d$ be this number. For example, if Alicia wrote down 1, 2, and 5, and Britt wrote down 2, 5, 7 , and 8 , then we have $a=3$ and $b=4$ while $v=5$ and $d=2$.
(a) Find an example for which $a=b=2022$ and $a \cdot b=d \cdot(v+d)$.
(b) Is is possible that $a \cdot b=d \cdot(v+4)$ ? Give an example or prove that it is impossible.
(c) Is it possible that $a \cdot b=d \cdot v$ ? Give an example or prove that it is impossible.

C2. We call a positive integer sunny if it has four digits and if moreover each of the two digits on the outside is exactly 1 larger than the digit next to it. The numbers 8723 and 1001 for example are sunny, but 1234 and 87245 are not.
(a) How many sunny numbers are there such that twice the number is again a sunny number?
(b) Prove that every sunny number greater than 2000 is divisible by a three-digit number with a 9 in the middle.

## Solutions

## B-problems

1. 11
2. 84
3. $2 \sqrt{6}$
4. 39
5. 20

## C-problems

$\mathbf{C 1}$. We first note that there is a useful relation between $a, b, d$, and $v$. The total number of integers on the two pieces of paper is $a+b$, the number of integers on Alicia's piece of paper plus the number of integers on Britt's piece of paper. This, however, also equals $v+d$ : the total number of distinct integers, plus the total number of integers that have been written down twice. Hence, we get that $a+b=v+d$.
(a) We choose $a=b=2022$ and look for a solution to $a \cdot b=d \cdot(v+d)$. We use the fact that $a+b=v+d$. This means that we are looking for solutions to $a \cdot b=d \cdot(a+b)$. If we substitute $a=b=2022$, then we find that $2022 \cdot 2022=d(2022+2022)=d \cdot 2 \cdot 2022$, so $d=1011$. Together with $a+b=v+d$, we find that $2022+2022=v+1011$, so $v=3033$. This situation happens for example if Alicia writes down the numbers 1 to 2022, and Britt writes down the numbers 1012 to 3033.
(b) With a little bit of trying, and by choosing $d$ not too large, we find that $a=b=3, d=1$, and $v=5$ is a solution: $3 \cdot 3=1 \cdot(5+4)$. The numbers also satisfy the equation $a+b=v+d$. This situation can occur if Alicia writes down the numbers 1,2 , and 3 , and Britt writes down the numbers 3,4 , and 5 , for example.
(c) Suppose that there are numbers such that $a \cdot b=v \cdot d$. We already deduced that $a+b=v+d$, or $v=a+b-d$. Substituting this yields

$$
a b=v d=(a+b-d) d=a d+b d-d^{2} .
$$

If we now subtract $a d$ from both sides of this equation, we find $a b-a d=b d-d^{2}$, so $a(b-d)=d(b-d)$. Because Britt wrote down
at least one number that Alicia did not write down, we have $b>d$. Therefore, we can divide the equation $a(b-d)=d(b-d)$ by the positive number $b-d$, and we find that $a=d$. On the other hand, Alicia wrote down at least one number that Britt did not write down, hence $a>d$. This gives a contradiction and hence there cannot exist numbers such that $a \cdot b=v \cdot d$.

C2. (a) First we look at the last two digits of a sunny number. There are nine possibilities for these: $01,12,23,34,45,56,67,78$, and 89 . If we then look at twice a sunny number, we get the following nine possibilities, respectively, for the last two digits: $02,24,46,68,90,12,34,56$, and 78. We see that twice a number can only be sunny if the original sunny number ends in $56,67,78$, or 89 . In all four cases we see that by doubling a 1 carries over to the hundreds.
Now we look at the first two digits of a sunny number. The nine possibilities are $10,21,32,43,54,65,76,87$, and 98 . If the first digit is 5 or higher, twice the number has more than four digits so it can never be sunny. The possibilities $10,21,32$, and 43 are left. After doubling and adding the carried over 1 to the hundreds we get, respectively, $21,43,65$, and 87 . In all cases twice a sunny number is a sunny number if the first digits of the original sunny number are $10,21,32$, or 43 and the last two digits are $56,67,78$, or 89 . In total there are $4 \cdot 4=16$ combinations to be made, hence 16 sunny numbers for which twice the number is again sunny.
(b) Denote by $a$ and $b$ the two middle digits of a sunny number. Then the two digits on the outside are $a+1$ and $b+1$, so the number is $1000(a+1)+100 a+10 b+(b+1)=1100 a+11 b+1001$. This number is divisible by 11 because $1100 a$ as well as $11 b$ as well as $1001=91 \cdot 11$ is divisible by 11. After division by 11 we get the number $100 a+b+91$. Now $b$ is at most 8 , because $b+1$ has to be a digit as well. Furthermore $a$ is at least 1 , because the number we started with has to be at least 2000. So we see that $100 a+b+91=100 a+10 \cdot 9+(b+1)$ is the three-digit number with digits $a, 9$, and $b+1$, a three-digit number with a 9 in the middle.

## Final Round, September 2022

## Problems

1. A positive integer $n$ is called divisor primary if for every positive divisor $d$ of $n$ at least one of the numbers $d-1$ and $d+1$ is prime. For example, 8 is divisor primary, because its positive divisors $1,2,4$, and 8 each differ by 1 from a prime number ( $2,3,5$, and 7 , respectively), while 9 is not divisor primary, because the divisor 9 does not differ by 1 from a prime number (both 8 and 10 are composite).
Determine the largest divisor primary number.
2. A set consisting of at least two distinct positive integers is called centenary if its greatest element is 100 . We will consider the average of all numbers in a centenary set, which we will call the average of the set. For example, the average of the centenary set $\{1,2,20,100\}$ is $\frac{123}{4}$ and the average of the centenary set $\{74,90,100\}$ is 88 .
Determine all integers that can occur as the average of a centenary set.
3. Given a positive integer $c$, we construct a sequence of fractions $a_{1}, a_{2}, a_{3}, \ldots$ as follows:

- $a_{1}=\frac{c}{c+1}$;
- to get $a_{n}$, we take $a_{n-1}$ (in its most simplified form, with both the numerator and denominator chosen to be positive) and we add 2 to the numerator and 3 to the denominator. Then we simplify the result again as much as possible, with positive numerator and denominator.

For example, if we take $c=20$, then $a_{1}=\frac{20}{21}$ and $a_{2}=\frac{22}{24}=\frac{11}{12}$. Then we find that $a_{3}=\frac{13}{15}$ (which is already simplified) and $a_{4}=\frac{15}{18}=\frac{5}{6}$.
(a) Let $c=10$, hence $a_{1}=\frac{10}{11}$. Determine the largest $n$ for which a simplification is needed in the construction of $a_{n}$.
(b) Let $c=99$, hence $a_{1}=\frac{99}{100}$. Determine whether a simplification is needed somewhere in the sequence.
(c) Find two values of $c$ for which in the first step of the construction of $a_{5}$ (before simplification) the numerator and denominator are divisible by 5 .
4. In triangle $A B C$, the point $D$ lies on segment $A B$ such that $C D$ is the angle bisector of angle $C$. The perpendicular bisector of segment $C D$ intersects the line $A B$ in $E$. Suppose that $|B E|=4$ and $|A B|=5$.

(a) Prove that $\angle B A C=\angle B C E$.
(b) Prove that $2|A D|=|E D|$.
5. Kira has 3 blocks with the letter A, 3 blocks with the letter B, and 3 blocks with the letter C. She puts these 9 blocks in a sequence. She wants to have as many distinct distances between blocks with the same letter as possible. For example, in the sequence ABCAABCBC the blocks with the letter A have distances 1,3 , and 4 between one another, the blocks with the letter B have distances 2,4 , and 6 between one another, and the blocks with the letter C have distances 2, 4, and 6 between one another. Altogether, we got distances of $1,2,3,4$, and 6 ; these are 5 distinct distances.
What is the maximum number of distinct distances that can occur?

## Solutions

1. Suppose $n$ is divisor primary. Then $n$ cannot have an odd divisor $d \geq 5$. Indeed, for such a divisor, both $d-1$ and $d+1$ are even. Because $d-1>2$, these are both composite numbers and that would contradict the fact that $n$ is divisor primary. The odd divisors 1 and 3 can occur, because the integer 3 itself is divisor primary.
Because of the unique factorisation in primes, the integer $n$ can now only have some factors 2 and at most one factor 3 . The number $2^{6}=64$ and all its multiples are not divisor primary, because both $63=7 \cdot 9$ and $65=5 \cdot 13$ are not prime. Hence, a divisor primary number has at most five factors 2 . Therefore, the largest possible number that could still be divisor primary is $3 \cdot 2^{5}=96$.

We now check that 96 is indeed divisor primary: its divisors are $1,2,3,4$, $6,8,12,16,24,32,48$, and 96 , and these numbers are next to $2,3,2,3,5$, $7,11,17,23,31,47$, and 97 , which are all prime. Therefore, the largest divisor primary number is 96 .
2. We solve this problem in two steps. First we will show that the smallest possible integral average of a centenary set is 14 , and then we will show that we can obtain all integers greater than or equal to 14 , but smaller than 100 , as the average of a centenary set.
If you decrease one of the numbers (unequal to 100) in a centenary set, the average becomes smaller. Also if you add a number that is smaller than the current average, the average becomes smaller. To find the centenary set with the smallest possible average, we can start with 1,100 and keep adjoining numbers that are as small as possible, until the next number that we would add is greater than the current average. In this way, we find the set with the numbers 1 to 13 and 100 with average $\frac{1}{14} \cdot(1+2+\ldots+13+100)=\frac{191}{14}=13 \frac{9}{14}$. Adding 14 would increase the average, and removing 13 (or more numbers) would increase the average as well. We conclude that the average of a centenary set must be at least 14 when it is required to be an integer.
Therefore, the smallest integer which could be the average of a centenary set is 14 , which could for example be realised using the following centenary set:

$$
\{1,2,3,4,5,6,7,8,9,10,11,12,18,100\}
$$

Now we still have to show that all integers greater than 14 (and smaller than 100) can indeed be the average of a centenary set. We start with the centenary set above with average 14 . Each time you add 14 to one of the numbers in this centenary set, the average increases by 1. Apply this addition from right to left, first adding 14 to 18 (the average becoming 15), then adding 14 to 12 (the average becoming 16), then adding 14 to 11 , et cetera. Then you end up with the centenary set

$$
\{15,16,17,18,19,20,21,22,23,24,25,26,32,100\}
$$

with average 27 , and you realised all values from 14 to 27 as an average. Because we started adding 14 to the second largest number in the set, this sequence of numbers remains increasing during the whole process, and therefore consists of 14 distinct numbers the whole time, and hence the numbers indeed form a centenary set.
We can continue this process by first adding 14 to 32 , then 14 to 26 et cetera, and then we get a centenary set whose average is 40 . Repeating this one more time, we finally end up with the set

$$
\{43,44,45,46,47,48,49,50,51,52,53,54,60,100\}
$$

with average 53. Moreover, we can obtain 54 as the average of the centenary set $\{8,100\}, 55$ as the average of $\{10,100\}$, and so on until 99 , which we obtain as the average of $\{98,100\}$. This shows that all values from 14 to 99 can be obtained.
3. (a) The sequence starts as follows.

$$
\begin{gathered}
a_{1}=\frac{10}{11}, \quad a_{2}=\frac{12}{14}=\frac{6}{7}, \quad a_{3}=\frac{8}{10}=\frac{4}{5}, \quad a_{4}=\frac{6}{8}=\frac{3}{4} \\
a_{5}=\frac{5}{7}, \quad a_{6}=\frac{7}{10}, \quad a_{7}=\frac{9}{13}
\end{gathered}
$$

It seems that the last simplification occurred at $a_{4}$. With induction to $n$, we will prove that there is no simplification for all $n \geq 5$. At the same time, we will prove that $a_{n}=\frac{1+2(n-3)}{1+3(n-3)}$ for all $n \geq 5$.
For $n=5$, the statement is true, because $a_{5}=\frac{5}{7}=\frac{1+2(5-3)}{1+3(5-3)}$ and this fraction $\frac{5}{7}$ cannot be simplified further. Now suppose the statement is true for $n=k-1$. Consider $n=k$. Because there has been no simplification for $a_{k-1}$, the numerator of $a_{k-1}$ equals $1+2(k-4)$ and the denominator equals $1+3(k-4)$. Then the number $a_{k}$ is defined as $\frac{1+2(k-4)+2}{1+3(k-4)+3}=\frac{1+2(k-3)}{1+3(k-3)}$.
We will argue by contradiction that there is no simplification here. Namely, suppose there is an integer $d>1$ such that both $1+2(k-3)$ and $1+3(k-3)$ are divisible by $d$. In particular, $3 \cdot(1+2(k-3))-2$. $(1+3(k-3))=1$ will also be divisible by $d$. This gives a contradiction, and the proof by induction is finished.
(b) We will show that there must be a simplification at some point. Indeed, suppose there is no simplification. Just like in part (a), we can show by induction that $a_{n}=\frac{97+2 n}{97+3 n}$. In particular, we see that $a_{97}$ is not a simplified fraction, because both the numerator and denominator are divisible by 97 , and that is a contradiction.
(c) You can use $c=7$ or $c=27$, for example. Then we get the sequences

$$
\frac{7}{8}, \quad \frac{9}{11}, \quad \frac{11}{14}, \quad \frac{13}{17}, \quad \frac{15}{20}=\frac{3}{4}
$$

and

$$
\frac{27}{28}, \quad \frac{29}{31}, \quad \frac{31}{34}, \quad \frac{33}{37}, \quad \frac{35}{40}=\frac{7}{8} .
$$

4. (a) In triangle $\triangle A D C$, the sum of the angles is $180^{\circ}$, hence

$$
\angle B A C=\angle D A C=180^{\circ}-\angle A D C-\angle A C D .
$$

Because $C D$ is the angle bisector of $\angle A C B$, we have $\angle A C D=\angle D C B$ and hence the equality above can be rewritten as

$$
\angle B A C=180^{\circ}-\angle A D C-\angle D C B
$$

Now we use that $\angle A D B$ is a straight angle, hence $\angle E D C=180^{\circ}-$ $\angle A D C$. Substituting this yields

$$
\angle B A C=\angle E D C-\angle D C B
$$

Because $E$ lies on the perpendicular bisector of $C D$, we have $\angle E D C=$ $\angle E C D$, and the equality becomes

$$
\angle B A C=\angle E C D-\angle D C B .
$$

Finally, we also see in the picture that $\angle E C D-\angle D C B=\angle B C E$, and hence

$$
\angle B A C=\angle B C E .
$$

(b) Triangles $\triangle A C E$ and $\triangle C B E$ are similar, because $\angle A E C=\angle C E B$ (same angle) and in part (a) we proved that $\angle B A C=\angle B C E$ and hence $\angle C A E=\angle B C E$. This yields

$$
\frac{|A E|}{|C E|}=\frac{|C E|}{|B E|}
$$

Using the fact that $|B E|=4$, we compute

$$
|A E|=|A B|+|B E|=5+4=9
$$

Substituting this in the ratios above, we obtain

$$
\frac{9}{|C E|}=\frac{|C E|}{4}
$$

hence $|C E|^{2}=36$ and $|C E|=6$. Because the perpendicular bisector of $C D$ passes through $E$, we have $|C E|=|D E|$. This yields

$$
6=|C E|=|D E|=|D B|+|B E|=|D B|+4
$$

and hence $|D B|=2$. Therefore, we conclude that

$$
|A D|=|A B|-|B D|=5-2=3 \quad \text { and } \quad|E D|=6 .
$$

We obtain that $2|A D|=2 \cdot 3=6=|E D|$.
5. We will show that the maximum number of distinct distances is 7 . First we prove that the number of distinct distances cannot be more than 7 , then we will show that there is a sequence of blocks with 7 distances.
The possible distances between two blocks in the sequence are the numbers 1 to 8 . Therefore, there can certainly be no more than 8 distinct distances. We will show that there is always at least one distance that does not occur. If in a sequence the distances 8 and 7 do not both occur, we are done. Therefore, suppose we have a sequence in which these two distances do both occur. The distance 8 can only occur between the very first and the very last block, so these should have the same letter on them, say A. The distance 7 can only occur between the first and the eighth (second last) block, or between the second and the last block. Because both outer blocks have an A, the second or eighth block must also have an A. Then the sequence of blocks is AAxxxxxxA (or the other way around: AxxxxxxAA), where on the place of $x$ are blocks with a B or C. Now we see that the distance 6 cannot occur anymore: the distances between the blocks with A are 1,7 , and 8 , and the distances between the blocks with $B$ and the blocks with C are at most 5 . Also in this case, there is at least one distance that does not occur.
We conclude that there is always one of the possible distances $1,2,3,4,5$, $6,7,8$ that does not occur. Hence, the number of distinct distances cannot be more than 7 .
An example of a sequence of blocks where 7 distinct distances occur, is ABBCACCBA, with distances $4,4,8 ; 1,5,6 ; 1,2,3$ (only the distance 7 is missing). So the maximal number of distinct distances is equal to 7 .

## BxMO Team Selection Test, March 2023

## Problems

1. Let $n \geq 1$ be an integer. Ruben takes a test with $n$ questions. Each question on this test is worth a different number of points. The first question is worth 1 point, the second question 2 , the third 3 and so on until the last question which is worth $n$ points. Each question can be answered either correctly or incorrectly. So an answer for a question can either be awarded all, or none of the points the question is worth. Let $f(n)$ be the number of ways he can take the test so that the number of points awarded equals the number of questions he answered incorrectly.
Do there exist infinitely many pairs $(a, b)$ with $a<b$ and $f(a)=f(b)$ ?
2. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ for which

$$
f(a-b) f(c-d)+f(a-d) f(b-c) \leq(a-c) f(b-d)
$$

for all real numbers $a, b, c$ and $d$.
Note that there is only one occurrence of $f$ on the right hand side!
3. We play a game of musical chairs with $n$ chairs numbered 1 to $n$. You attach $n$ leaves, numbered 1 to $n$, to the chairs in such a way that the number on a leaf does not match the number on the chair it is attached to. One player sits on each chair. Every time you clap, each player looks at the number on the leaf attached to his current seat and moves to sit on the seat with that number. Prove that, for any $m$ that is not a prime power with $1<m \leq n$, it is possible to attach the leaves to the seats in such a way that after $m$ claps everyone has returned to the chair they started on for the first time.
4. In a triangle $\triangle A B C$ with $\angle A B C<\angle B C A$, we define $K$ as the excentre with respect to $A$. The lines $A K$ and $B C$ intersect in a point $D$. Let $E$ be the circumcentre of $\triangle B K C$. Prove that

$$
\frac{1}{|K A|}=\frac{1}{|K D|}+\frac{1}{|K E|} .
$$

5. Find all pairs of prime numbers $(p, q)$ for which

$$
2^{p}=2^{q-2}+q!
$$

## Solutions

1. For the first few values of $f$, note that $f(1)=0, f(2)=f(3)=f(4)=1$, $f(5)=f(6)=2, f(7)=f(8)=3, f(9)=f(10)=5$. We claim that for $n \geq 11, f(n)$ is strictly increasing as a function of $n$. Therefore there is only a finite number of pairs as in the problem.
We view a way of taking a test as a subset of $\{1,2, \ldots, n\}$ by taking the set of numbers of questions that are answered correctly. We say that a subset $S_{n}$ of $\{1,2, \ldots, n\}$ is an $n$-equally correct set if the sum of all elements of $S_{n}$ is equal to the number of elements in the complement of $S_{n}$ in $\{1,2, \ldots, n\}$. So by definition, $f(n)$ is the number of $n$-equally correct sets. Note that a subset $S_{n}$ of $\{1,2, \ldots, n\}$ is $n$-equally correct if and only if the sum of all elements of $S_{n}$ plus the number of elements of $S_{n}$ equals $n$.
We first claim that $f(n)$ is non-decreasing for $n \geq 1$. Let $S_{n-1}$ be an $(n-1)$-equally correct set. Then adding 1 to the largest element of $S_{n-1}$ gets us a subset $S_{n}$ of $\{1,2, \ldots, n\}$. This subset has as many elements as $S_{n-1}$ and has sum 1 higher than the sum of $S_{n-1}$, which is therefore an $n$-equally correct set. This procedure defines an injective map $F_{n}$ from the set of $(n-1)$-equally correct sets to the set of $n$-equally correct sets for all $n \geq 2$. Therefore $f(n) \geq f(n-1)$ for all $n \geq 2$.
We now show that, for $n \geq 11$, there exists an $n$-equally correct set that is not in the image of $F_{n}$. Note that an $n$-equally correct set of which the two largest elements differ by exactly 1 cannot be in the image of $F_{n}$; the reason is that a set in this image is one that is obtained by adding 1 to the largest element of a subset of $\{1,2, \ldots, n-1\}$. If $n=2 k+1$ with $k \geq 5$, then $S=\{1, k-2, k-1\}$ is $n$-equally correct, as $1+(k-2)+(k-1)+|S|=$ $2 k+1=n$. Similarly, if $n=2 k$ with $k \geq 6$, then $S=\{2, k-3, k-2\}$ is $n$-equally correct, because $2+(k-3)+(k-2)+|S|=2 k=n$. We conclude that $f(n)>f(n-1)$ for all $n \geq 11$.
2. The solutions to the given functional inequality are $f(x)=0$ for all $x$ and $f(x)=x$ for all $x$. For $f(x)=0$, we easily find that equality always holds. For $f(x)=x$ we check that

$$
\begin{aligned}
(a-b)(c-d)+(a-d)(b-c) & =a c-a d-b c+b d+a b-a c-b d+c d \\
& =-a d-b c+a b+c d \\
& =(a-c)(b-d)
\end{aligned}
$$

so equality holds in that case too. Now we show that these functions are the only two solutions.

Substituting $a=b=c=d=0$ gives us $2 f(0)^{2} \leq 0$, and therefore $f(0)=0$. Then we substitute $b=a-x, c=a$, and $d=a-y$, which gives $a-b=x$, $a-c=0$ and $a-d=y$. From this we deduce that

$$
\begin{equation*}
f(y)(f(x)+f(-x)) \leq 0 \tag{1}
\end{equation*}
$$

Suppose there is a $y$ such that $f(y) \neq 0$. If we then substitute $x=y$ in the equation above and move one of the terms to the right, we find that

$$
0<f(y)^{2} \leq-f(y) f(-y)
$$

Therefore one of the two values $f(y)$ and $f(-y)$ is positive and the other is negative. Assume without loss of generality that $f(y)$ is positive.
Now, given arbitrary $a$ and $y$, substitute $b=a, c=0$, and $d=a-y$. Then we find that $f(y) f(a) \leq a f(y)$. Thus, if we divide both sides by the positive $f(y)$, we get $f(a) \leq a$. On the other hand, if we substitute $b=a, c=0$, and $d=a+y$, we find that $f(-y) f(a) \leq a f(-y)$. Since $f(-y)$ is negative, the sign flips when we divide by $f(-y)$ so we deduce that $f(a) \geq a$. We conclude that $f(a)=a$ for all real $a$.
Therefore a solution $f$ of the given functional inequality is either the zero function or $f(a)=a$ for all real $a$, and we confirmed in the beginning that both are indeed solutions.
3. If $m=n$, then attach to chair $i$ the leaf with number $i+1$. Everyone then moves up a chair every clap and for everyone, the first time they return to the chair they started on is after $n$ claps. So after $n$ claps for the first time, everyone has returned to the chair they started on. Now suppose $m<n$.
Since $m$ is not a prime power, we can write $m$ as $m=k \ell$ with $\operatorname{gcd}(k, \ell)=1$. We claim that there are $a, b>0$ such that

$$
a k+b \ell=n .
$$

Note that the numbers $n-a k$ with $a \in\{1,2, \ldots, \ell\}$ are all different modulo $\ell$. We show this by contraposition. Suppose that there are $a_{1}$ and $a_{2}$ in $\{1,2, \ldots, \ell\}$ such that $n-a_{1} k \equiv n-a_{2} k \bmod \ell$. Then we also have $\left(a_{1}-a_{2}\right) k \equiv 0 \bmod \ell$. As $\operatorname{gcd}(k, \ell)=1$, it follows that $\ell \mid\left(a_{1}-a_{2}\right)$. Since $a_{1}$ and $a_{2}$ both are in $\{1,2, \ldots, \ell\}$, it follows that $a_{1}=a_{2}$. Therefore if $a_{1}$ and $a_{2}$ are in $\{1,2, \ldots, \ell\}$ and $a_{1} \neq a_{2}$, then $n-a_{1} k \not \equiv n-a_{2} k \bmod \ell$. So the numbers $n-a k$ with $a \in 1,2, \ldots, \ell$ are indeed all different modulo $\ell$.
It follows that the $n-a k$ for $a \in\{1,2, \ldots, \ell\}$ are all the $\ell$ residue classes modulo $\ell$. Since $n-a k \geq n-\ell k=n-m>0$, these $\ell$ numbers are also
all greater than 0 . Now choose the $a$ for which $n-a k$ is congruent to 0 modulo $\ell$. Since $n-a k$ is greater than 0 , there is now a $b>0$ such that $n-a k=b \ell$. This gives the $a, b>0$ such that $a k+b \ell=n$.
Now we divide the chairs into $a$ groups of $k$ chairs and $b$ groups of $\ell$ chairs. In each group, we arrange the chairs in a circle and attach the leaves to chairs in such a way that each leaf has the number of the next chair in the circle. The players on a chair in a group with $k$ chairs, return to the chair they started on every $k$ claps (and not on any other clap). The players on a seat in a group with $\ell$ seats, return to the chair they started on every $\ell$ claps (and not on any other clap). So the first time everyone returns to the chair they started on is after $\operatorname{lcm}(k, \ell)$ claps. But

$$
\operatorname{lcm}(k, \ell)=k \ell=m
$$

because $\operatorname{gcd}(k, \ell)=1$.

4. Write $\angle C A B=2 \alpha, \angle A B C=2 \beta$ and $\angle B C A=2 \gamma$. Note that $\alpha+\beta+\gamma=$ $\frac{1}{2}(\angle C A B+\angle A B C+\angle B C A)=\frac{1}{2} \cdot 180^{\circ}=90^{\circ}$.
We first prove that $K, A$ and $E$ are collinear. Since $K$ lies on the interior bisector of $\angle A B C$ and on the exterior bisector of angle $\angle C A B$ we find that $\angle B K A=180^{\circ}-\angle K A B-\angle A B K=180^{\circ}-\left(90^{\circ}+\alpha\right)-\beta=90^{\circ}-\alpha-\beta=\gamma$. On the other hand, we know that $\angle B C K=90^{\circ}+\gamma>90^{\circ}$ is obtuse. Since $E$ is the circumcentre of $\triangle B K C, E$ therefore lies on the opposite side of $B K$ to $C$, and using the inscribed angle theorem, we find that $\angle K E B=2 \cdot\left(180^{\circ}-\angle B C K\right)=2 \cdot\left(90^{\circ}-\gamma\right)=180^{\circ}-2 \gamma$. However, since $E$ is the circumcentre of $\triangle B K C$, we also know that $\triangle B E K$ is isosceles with apex $E$. So $\angle B K E=\frac{1}{2}\left(180^{\circ}-\angle K E B\right)=\frac{1}{2} \cdot 2 \gamma=\gamma$. We deduce that $\angle B K A=\gamma=\angle B K E$, from which it follows that $K, A$ and $E$ are collinear.
Because of this collinearity, we find that $\angle A E B=\angle K E B=180^{\circ}-2 \gamma=$ $180^{\circ}-\angle B C A$, so $A C B E$ is a cyclic quadrilateral. Therefore $\angle A E C=$
$\angle A B C=\angle A B D$. Since also $\angle C A E=90^{\circ}+\alpha=\angle K A B=\angle D A B$, we find that $\triangle A E C \sim \triangle A B D$ due to equal angles.
The last two observations we need are that $|E C|=|E K|$ because $\triangle C E K$ is also isosceles with apex $E$, and that $B K$ is an internal angle bisector of $\triangle A B D$. Now it follows that

$$
1-\frac{|K A|}{|K E|}=\frac{|K E|-|K A|}{|K E|}=\frac{|A E|}{|K E|}=\frac{|A E|}{|C E|}=\frac{|A B|}{|D B|}=\frac{|A K|}{|D K|},
$$

where we have used the following respectively: the difference of fractions, the collinearity of $K, A$, and $E$, the equality $|E C|=|E K|$, the similarity $\triangle A E C \sim \triangle A B D$, and the angle bisector theorem on $B K$ in $\triangle A B D$. The required property now follows simply by taking $\frac{|K A|}{|K E|}$ to the other side and dividing by $|K A|$.
5. Answer: the only pairs $(p, q)$ that satisfy the given condition are $(3,3)$ and $(7,5)$.
First, we check a few cases for small $q$. If $q=2$, then $2^{p}=1+2$ has no solution. If $q=3$, then $2^{p}=2+6$ gives that $p=3$ is the only solution. If $q=5$, then $2^{p}=8+120$ gives that $p=7$ is the only solution. We claim that there are no solutions with $q \geq 7$, and prove this by contradiction.
Let $q \geq 7$ be prime, and let $p$ be a prime such that $(p, q)$ satisfies the given condition. Now we note that we can rewrite the given condition as

$$
\begin{equation*}
q!=2^{p}-2^{q-2}=2^{q-2}\left(2^{p-q+2}-1\right) . \tag{2}
\end{equation*}
$$

Since $q$ ! is positive, so is the right-hand side. In particular, $2^{p-q+2}>1$, so $p-q+2>0$ and $2^{p-q+2}$ is integer (and greater than 1 ). We conclude that $2^{p-q+2}-1$ is integer and odd, therefore the right-hand side has exactly $q-2$ factors 2 . To count the number of factors on the left-hand side let $\nu_{2}(a)$ be the function that counts how many factors 2 an integer $a$ has. Since $q$ is a prime number greater than 2 , it is odd. Then $\left\lfloor\frac{q}{2}\right\rfloor=\frac{q-1}{2}$ and in general $\left\lfloor\frac{q}{2^{i}}\right\rfloor=\left\lfloor\frac{q-1}{2^{i}}\right\rfloor$. Now let $n$ be the integer such that $2^{n}<q<2^{n+1}$. Then we compute

$$
\begin{aligned}
\nu_{2}(q!) & =\left\lfloor\frac{q}{2}\right\rfloor+\left\lfloor\frac{q}{4}\right\rfloor+\ldots+\left\lfloor\frac{q}{2^{n}}\right\rfloor \\
& =\left\lfloor\frac{q-1}{2}\right\rfloor+\left\lfloor\frac{q-1}{4}\right\rfloor+\ldots+\left\lfloor\frac{q-1}{2^{n}}\right\rfloor \\
& \leq(q-1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n}}\right) \\
& =(q-1)\left(1-\frac{1}{2^{n}}\right) \\
& =(q-1)-\frac{q-1}{2^{n}} \\
& \leq q-2 .
\end{aligned}
$$

In both inequalities above, equality holds if and only if $q=2^{n}+1$ (and we have thus used in the second line that $q$ is odd). As the number of factors in the right-hand side of $(2)$ is also $q-2$, equality must hold, so we must have $q=2^{n}+1$ for a certain positive integer $n$.
Now that we have come a long way by looking at factors 2 , we will now derive the desired contradiction from equation (2) with modular arithmetic, using the fact that $q$ ! has many prime factors. From our small examples, we know this is not going to work modulo 3 or 5 , so try the next prime number: 7 .

Since we have assumed that $q$ is at least $7, q$ ! is indeed divisible by 7 . Therefore we have $2^{p-q+2} \equiv 1 \bmod 7$. Since also $2^{3} \equiv 8 \equiv 1 \bmod 7$, it follows that $2^{\operatorname{gcd}(p-q+2,3)} \equiv 1 \bmod 7$. But $2^{1} \not \equiv 1 \bmod 7$, so $\operatorname{gcd}(p-q+$ $2,3)=3$, hence $p-q+2 \equiv 0 \bmod 3$. However, we know that $q=2^{n}+1$ is congruent to -1 or 0 modulo 3 . Since $q$ is prime (and greater than 3 ), the second case cannot occur and we conclude that $q \equiv-1 \bmod 3$. But then it follows from $p-q+2 \equiv 0 \bmod 3$ that $p \equiv q-2 \equiv-1-2 \equiv 0 \bmod 3$. Since $p$ is also prime, we conclude that $p=3$. But then $2^{q-2}+q!=2^{3}=8$ has no solutions for $q \geq 7$.
We conclude that the only pairs $(p, q)$ that satisfy the given condition are $(3,3)$ and $(7,5)$.

## IMO Team Selection Test 1, June 2023

## Problems

1. Let $\triangle A B C$ be a triangle with $|A B|<|A C|<|B C|$ and with circumcircle $\Gamma$ having centre $O$. Let $\omega_{1}$ be the circle with centre $B$ and radius $|A C|$ and let $\omega_{2}$ be the circle with centre $C$ and radius $|A B|$. The circles $\omega_{1}$ and $\omega_{2}$ intersect in a point $E$ such that $A$ and $E$ lie on opposite sides of the line $B C$. The circles $\Gamma$ and $\omega_{1}$ intersect in a point $F$ and the circles $\Gamma$ and $\omega_{2}$ intersect in a point $G$ such that $F$ and $G$ lie on the same side of the line $B C$ as $E$.
Prove that the antipode $K$ of $A$ relative to $\Gamma$ is the circumcentre of $\triangle E F G$.
2. Determine the largest real number $M$ such that for each infinite sequence $x_{0}, x_{1}, x_{2}, \ldots$ of real numbers satisfying
(a) $x_{0}=1$ and $x_{1}=3$,
(b) $x_{0}+x_{1}+\cdots+x_{n-1} \geq 3 x_{n}-x_{n+1}$ for all $n \geq 1$,
the inequality

$$
\frac{x_{n+1}}{x_{n}}>M,
$$

holds for all $n \geq 0$.
3. Find all positive integers $n$ for which there exist $n$ distinct positive integers $a_{1}, a_{2}, \ldots, a_{n}$, none of them greater than $n^{2}$, such that

$$
\begin{equation*}
\frac{1}{a_{1}}+\frac{1}{a_{2}}+\cdots+\frac{1}{a_{n}}=1 . \tag{3}
\end{equation*}
$$

4. Given a positive integer $n$ define $\tau(n)$ as the number of positive divisors of $n$, and define $\sigma(n)$ as the sum of these divisors. Find all positive integers $n$ for which

$$
\sigma(n)=\tau(n) \cdot\lceil\sqrt{n}\rceil .
$$

For a real number $x$, we use the notation $\lceil x\rceil$ to mean the smallest integer greater than or equal to $x$.

## Solutions

1. Since $|B E|=|A C|$ and $|C E|=|A B|, A B E C$ is a parallelogram. Therefore $B E \| A C$. On the other hand, $A B G C$ is a cyclic quadrilateral with $|C G|=|A B|$. It follows that $A B G C$ is an isosceles trapezoid with $B G \| A C$. Therefore $B, G$, and $E$ are collinear. Completely analogously, we see that $C, F$, and $E$ are collinear.
Since $K$ is the antipode of $A$ relative to $\Gamma$, we have $\angle A C K=90^{\circ}$. As $A C \| B E$ and $G$ lies on the line $B E$, it follows that $C K$ is perpendicular to $G E$. But we also know that $|C E|=|A B|=|C G|$, so $\triangle E C G$ is isosceles and $C K$ is the perpendicular bisector of $E G$. In the same way, we see that $B K$ is the perpendicular bisector of $E F$. So $K$ lies on the perpendicular bisectors of $E G$ and $E F$. We conclude that $K$ is the circumcentre of $\triangle E F G$.

2. Answer: the largest possible $M$ for which the given property holds is $M=2$. We first show that the given property holds for $M=2$. To do this, we show by induction on $n$ the stronger statement that $x_{n+1}>2 x_{n}>x_{n}+x_{n-1}+$ $\ldots+x_{0}$ for all $n \geq 0$. For $n=0$, this is the statement $x_{1}>2 x_{0}>x_{0}$ which translates with the given initial values to $3>2>1$. Now suppose for the induction hypothesis that $x_{n+1}>2 x_{n}>x_{n}+x_{n-1}+\ldots+x_{0}$. Then we find for $x_{n+2}$ :

$$
\begin{aligned}
x_{n+2} & \geq 3 x_{n+1}-\left(x_{n}+\ldots+x_{0}\right) \\
& >2 x_{n+1} \\
& >x_{n+1}+x_{n}+\ldots+x_{0} .
\end{aligned}
$$

This completes the induction step. By induction, it follows that for all sequences $x$ satisfying (a) and (b), the inequality $\frac{x_{n+1}}{x_{n}}>2$ holds for all $n \geq 0$.

To show that we cannot find a higher value for $M$, we look at the sequence $x$ with $x_{0}=1, x_{1}=3$, and for which equality holds in (b), i.e. $x_{0}+x_{1}+$ $\cdots+x_{n-1}=3 x_{n}-x_{n+1}$ for all $n \geq 1$. Then the following relation holds:

$$
\begin{aligned}
x_{n+1} & =3 x_{n}-\left(x_{n-1}+\ldots+x_{0}\right) \\
& =3 x_{n}-x_{n-1}-\left(x_{n-2}+\ldots+x_{0}\right) \\
& =3 x_{n}-x_{n-1}-\left(3 x_{n-1}-x_{n}\right) \\
& =4 x_{n}-4 x_{n-1} .
\end{aligned}
$$

Note that this is a homogeneous linear recurrence relation, the characteristic equation of which is $\lambda^{2}-4 \lambda+4=(\lambda-2)^{2}=0$. Since the characteristic equation has a double root at $\lambda=2$, the general solution of the recurrence relation is of the form $x_{n}=B 2^{n}+C n 2^{n}$ for real numbers $B$ and $C$. If we now solve this for the given starting values $x_{0}=1$ and $x_{1}=3$, we get the system of equations $B+0=x_{0}=1$ and $2 B+2 C=x_{1}=3$. Its unique solution is given by $B=1$ and $C=\frac{1}{2}$. So the solution for these starting values is $x_{n}=1 \cdot 2^{n}+\frac{1}{2} n 2^{n}=(n+2) 2^{n-1}$.
Now that we have solved the recurrence relation, a simple computation yields

$$
\frac{x_{n+1}}{x_{n}}=\frac{(n+3) 2^{n}}{(n+2) 2^{n-1}}=2 \frac{n+3}{n+2}=2\left(1+\frac{1}{n+2}\right) .
$$

So for sufficiently large $n$, this fraction becomes arbitrarily close to 2 . To state this more precisely: suppose $M=2+\varepsilon$ with $\varepsilon>0$. Then, for this sequence and $n>\frac{2}{\varepsilon}-2$, we have $\frac{x_{n+1}}{x_{n}}=2+\frac{2}{n+2}<2+\varepsilon=M$. Therefore no such $M$ can have the given property. So the largest value of $M$ that has the given property is 2 , and in the first part of this solution we have already seen that $M=2$ has the given property.
3. The answer is that the given property holds for all $n \neq 2$. For $n=1$, the set $\{1\}$ satisfies (3). For $n=2$, note that no set satisfies (3); if $a_{1}$ or $a_{2}$ equals 1, then $\frac{1}{a_{1}}+\frac{1}{a_{2}}>1$, if $a_{1}$ and $a_{2}$ are both at least two, then $\frac{1}{a_{1}}+\frac{1}{a_{2}} \leq \frac{1}{2}+\frac{1}{3}<1$.
So now suppose that $n \geq 3$. Note that we have the identity

$$
\begin{equation*}
\frac{1}{k}=\frac{1}{k+\ell}+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}+\cdots+\frac{1}{(k+\ell-1)(k+\ell)} . \tag{4}
\end{equation*}
$$

Indeed, if we use $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$ then the right-hand side of (4) is equal to

$$
\frac{1}{k+\ell}+\left(\frac{1}{k}-\frac{1}{k+1}\right)+\left(\frac{1}{k+1}-\frac{1}{k+2}\right)+\cdots+\left(\frac{1}{k+\ell-1}-\frac{1}{k+\ell}\right)=\frac{1}{k+\ell}+\frac{1}{k}-\frac{1}{k+\ell} .
$$

Substituting $k=1$ and $\ell=n-1$ into (4), we obtain an identity

$$
\begin{equation*}
1=\frac{1}{n}+\frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+\frac{1}{3 \cdot 4}+\frac{1}{4 \cdot 5}+\cdots+\frac{1}{(n-1) \cdot n} . \tag{5}
\end{equation*}
$$

If $n \neq k(k+1)$ for all $k \geq 1$, this is a sum of $n$ distinct reciprocals, each with denominator smaller than $n^{2}$. This shows that the given property holds for all $n \geq 3$ not of the form $k(k+1)$.
Suppose that there exists some $k \geq 1$ such that $n=k(k+1)$. Then we apply to the right-hand side of (5) the substitutions $\frac{1}{n}+\frac{1}{(n-1) n}=\frac{1}{n-1}$ and $\frac{1}{6}=\frac{1}{10}+\frac{1}{15}$. Then we get the identity

$$
\begin{equation*}
1=\frac{1}{n-1}+\frac{1}{2}+\frac{1}{10}+\frac{1}{15}+\frac{1}{3 \cdot 4}+\frac{1}{4 \cdot 5}+\cdots+\frac{1}{(n-2) \cdot(n-1)} . \tag{6}
\end{equation*}
$$

This is a sum of $n$ reciprocals. Note that each denominator of each reciprocal is smaller than $n^{2}$; this is only non-obvious for the term $\frac{1}{15}$, and since $n=k(k+1)$ and $n \geq 3$, we in particular have $n \geq 6$, and therefore $n^{2}>15$. Now we show that these reciprocals are distinct.
Note that $k(k+1)$ is always even. Since $n$ is of the form $k(k+1), n-1$ is odd and therefore not of this form. Furthermore, $n-1$ is also unequal to 2,10 and 15 , since 3,11 and 16 are not of the form $k(k+1)$. Also, 10 and 15 themselves are not of the form $k(k+1)$. Therefore all the reciprocals in the right-hand side of (6) are distinct. So the given property holds for all $n \geq 3$ of the form $k(k+1)$ as well.
4. Answer: the solutions for $n$ are $n=1,3,5,6$.

Indeed, for these four cases we have respectively, $1=1 \cdot 1,4=2 \cdot\lceil\sqrt{3}\rceil$, $6=2 \cdot\lceil\sqrt{5}\rceil$, and $12=4 \cdot\lceil\sqrt{6}\rceil$. From now on, we assume that $n \neq 1$.
If $n$ is square, then $n$ has an odd number of divisors $1=d_{1}<d_{2}<\ldots<$ $d_{2 k-1}=n$. The middle of these is $d_{k}=\sqrt{n}$; all other divisors can be partitioned into $k-1$ pairs $\left(d_{i}, d_{2 k-i}\right)$ with $d_{i} d_{2 k-i}=n$. Using the AM-GM inequality, we then find that

$$
\frac{\sigma(n)}{\tau(n)}=\frac{\sum_{i=1}^{2 k-1} d_{i}}{2 k-1}>\sqrt[2 k-1]{\prod_{i=1}^{2 k-1} d_{i}}=\sqrt[2 k-1]{\sqrt{n} \cdot n^{k-1}}=\sqrt{n}=\lceil\sqrt{n}\rceil,
$$

Note that in the instance of the AM-GM inequality used, we cannot have equality because $d_{1} \neq d_{2 k-1}$. Therefore for $n$ square there are no solutions. If $n$ is not square, then $n$ has an even number of divisors. We can see this by taking the prime factorisation $n=p_{1}^{e_{1}} p_{2}^{e_{2}} \cdots p_{t}^{e_{t}}$; we then have $\tau(n)=\left(e_{1}+1\right)\left(e_{2}+1\right) \cdots\left(e_{t}+1\right)$ and at least one of $e_{i}$ must be odd as $n$ is
not square. We again order the divisors by size $1=d_{1}<d_{2}<\ldots<d_{2 k}=n$. For the middle two of these divisors, we find

$$
\frac{d_{k}+d_{k+1}+1}{2} \geq \frac{d_{k}+d_{k+1}}{2} \geq \sqrt{d_{k} d_{k+1}}=\sqrt{n}
$$

Since either $\frac{d_{k}+d_{k+1}+1}{2}$ or $\frac{d_{k}+d_{k+1}}{2}$ is integer, it follows that $\frac{d_{k}+d_{k+1}+1}{2}$ is greater than or equal to a positive integer greater than or equal to $\sqrt{n}$. We deduce that $\frac{d_{k}+d_{k+1}+1}{2} \geq\lceil\sqrt{n}\rceil$, which we can rewrite as $d_{k}+d_{k+1} \geq$ $2\lceil\sqrt{n}\rceil-1$.
Suppose we have two divisors $d<e \leq \sqrt{n}$ of $n$. Then both sequences $\left(d, \frac{n}{e}\right)$ and $\left(1, \frac{e}{d}\right)$ are strictly increasing, because $d<\sqrt{n} \leq \frac{n}{e}$ and $1<\frac{e}{d}$, respectively. Using the rearrangement inequality, we then find that $d+\frac{n}{d}>$ $e+\frac{n}{e}$. Note that this is a strict inequality because both rows are strictly increasing. With $e=d_{k}$ we then find that $d+\frac{n}{d}>d_{k}+d_{k+1} \geq 2\lceil\sqrt{n}\rceil-1$. So $d_{k-i}+d_{k+1+i}>2\lceil\sqrt{n}\rceil-1$ and therefore $d_{k-i}+d_{k+1+i} \geq 2\lceil\sqrt{n}\rceil$ for all $1 \leq i \leq k-1$.
Now suppose that $n \geq 8$. Then note that $(n-4)^{2} \geq 4^{2}>12$, and therefore that $n^{2}-8 n+16 \geq 12$. We can also write this as $n^{2}-4 n+4 \geq 4 n$ and then decompose this as $(n-2)^{2} \geq 4 n$. So taking the square root of both sides of this inequality, we find that

$$
\frac{n-1}{2} \geq \frac{n-2}{2} \geq \sqrt{n}
$$

As before, we note that $\frac{n-1}{2}$ is now greater than or equal to a positive integer greater than or equal to $\sqrt{n}$, so $\frac{n-1}{2} \geq\lceil\sqrt{n}\rceil$. We can rewrite this as $n+1 \geq 2\lceil\sqrt{n}\rceil+2$.
For non-square $n \geq 8$ we conclude that

$$
\begin{aligned}
\sigma(n) & =\left(d_{k}+d_{k+1}\right)+\left(d_{k-1}+d_{k+2}\right)+\ldots+\left(d_{2}+d_{2 k-1}\right)+\left(d_{1}+d_{2 k}\right) \\
& \geq(2\lceil\sqrt{n}\rceil-1)+2\lceil\sqrt{n}\rceil+\ldots+2\lceil\sqrt{n}\rceil+(2\lceil\sqrt{n}\rceil+2) \\
& =2 k\lceil\sqrt{n}\rceil+1 \\
& =\tau(n) \cdot\lceil\sqrt{n}\rceil+1
\end{aligned}
$$

so no such $n$ can be a solution.
Finally none of the remaining cases can be solutions, since for $n=2$ we have $3 \neq 2 \cdot\lceil\sqrt{2}\rceil$ and for $n=7$ we have $8 \neq 2 \cdot\lceil\sqrt{7}\rceil$. (The case $n=4$ was handled in the square case.)

## IMO Team Selection Test 2, June 2023

## Problems

1. Let $n$ be a positive integer. Prove that the numbers

$$
1^{1}, 3^{3}, 5^{5}, \ldots,\left(2^{n}-1\right)^{2^{n}-1}
$$

are in different residue classes modulo $2^{n}$.
2. A triangle $A B C$ and a point $D$ on the line segment $A C$ are given. Let $M$ be the midpoint of $C D$ and let $\Omega$ be the circle through $B$ and $D$ tangent to $A B$. Let $E$ be the point such that $\triangle M D B \sim \triangle M B E$ and such that $D$ and $E$ lie on opposite sides of the line $M B$.
Show that $E$ lies on $\Omega$ if and only if $\angle A B D=\angle M B C$.
3. Find the smallest possible value of

$$
x y+y z+z x+\frac{1}{x}+\frac{2}{y}+\frac{5}{z}
$$

with $x, y, z$ positive real numbers.
4. Let $n \geq 3$ be a fixed positive integer. There are $n$ boxes $A_{1}, A_{2}, \ldots, A_{n}$, each with a number of stones in it $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ such that $a_{1}+a_{2}+\cdots+a_{n}=3 n$. A move consists of the following operations:
choose a box and distribute all the stones in the box among the $n$ boxes (including the box that was chosen) such that for every two boxes the numbers of stones added to those boxes differ by at most 1 .

For a distribution $a_{1}, a_{2}, \ldots, a_{n}$, we define $f\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ as the least number of moves required to get all the stones into a single box. Let $M_{n}$ be the maximum of $f\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ for all possible distributions $a_{1}, a_{2}, \ldots, a_{n}$ such that $a_{1}+a_{2}+\cdots+a_{n}=3 n$. Determine $M_{n}$ and all distributions $a_{1}, a_{2}, \ldots, a_{n}$ for which $f\left(a_{1}, a_{2}, \ldots, a_{n}\right)=M_{n}$.
Example. If $n=4$ and the boxes contain 2, 6, 0, 4 stones in that order, then we can distribute the 2 stones from box $A_{1}$ by putting in each box in order 1, 0, 1, 0 stones. After this move, the number of stones in each box in order is 1, 6, $1,4$.

## Solutions

1. We proceed by induction on $n$. For $n=1$, the only number in the sequence is $1^{1}$, so the given statement is trivially true.
For the induction step, we are to show that for $n \geq 1$, the numbers $1^{1}, 3^{3}, 5^{5}, \ldots,\left(2^{n+1}-1\right)^{2^{n+1}-1}$ are in different residue classes modulo $2^{n+1}$, given the induction hypothesis that $1^{1}, 3^{3}, 5^{5}, \ldots,\left(2^{n}-1\right)^{2^{n}-1}$ are in different residue classes modulo $2^{n}$.
So suppose that that is the case. We split the numbers $1^{1}, 3^{3}, 5^{5}, \ldots,\left(2^{n+1}-\right.$ $1)^{2^{n+1}-1}$ into two groups, namely $1^{1}, 3^{3}, 5^{5}, \ldots,\left(2^{n}-1\right)^{2^{n}-1}$, which we will call the lesser group, and $\left(2^{n}+1\right)^{2^{n}+1},\left(2^{n}+3\right)^{2^{n}+3}, \ldots,\left(2^{n+1}-1\right)^{2^{n+1}-1}$, which we will call the greater group. First note that the numbers in the lesser group are also all in different residue classes modulo $2^{n+1}$.
Note that because $\varphi\left(2^{n+1}\right)=2^{n}$, we have $a^{k} \equiv a^{\ell} \bmod 2^{n+1}$ for $k, \ell, a$ such that $k \equiv \ell \bmod 2^{n}$ and $a$ is odd. We will use this observation to compare the residue classes of the greater group to those of the lesser group. Write the numbers in the greater group as $\left(2^{n}+m\right)^{2^{n}+m}$ for $1 \leq m \leq 2^{n}-1$ and odd. Expanding $\left(2^{n}+m\right)^{2^{n}+m}$ using Newton's binomial, we note that any term with at least two factors $2^{n}$ is congruent to 0 modulo $2^{n+1}$. We thus find modulo $2^{n+1}$ that

$$
\begin{align*}
\left(2^{n}+m\right)^{2^{n}+m} & \equiv m^{2^{n}+m}+\left(2^{n}+m\right) m^{2^{n}+m-1} 2^{n}+2^{2 n}(\cdots) \\
& \equiv m^{2^{n}+m}+\left(2^{n}+m\right) m^{2^{n}+m-1} 2^{n} \\
& \equiv m^{m}+\left(2^{2 n}+2^{n} m\right) m^{m-1} \\
& \equiv m^{m}+2^{n} \cdot m^{m} \\
& \equiv m^{m}+2^{n} \tag{7}
\end{align*}
$$

where we have used in the last step that $m^{m}$ is odd; writing $m^{m}$ as $2 a+1$ shows that $2^{n}(2 a+1)=2^{n+1} a+2^{n} \equiv 2^{n} \bmod 2^{n+1}$.
Since the numbers $m^{m}$ from the lesser group are different modulo $2^{n+1}$, the numbers from the greater group therefore are also different modulo $2^{n+1}$. Moreover, from (7) it follows that the numbers from the lesser group are different from those from the greater group modulo $2^{n+1}$. Indeed, suppose for a contradiction that $\left(2^{n}+m\right)^{2^{n}+m} \equiv k^{k} \bmod 2^{n+1}$ with $1 \leq k, m \leq 2^{n-1}$, then it follows from (7) that $m^{m} \equiv m^{m}+2^{n} \equiv k^{k}$ $\bmod 2^{n}$. So because of the induction hypothesis, it follows that $m=k$. But in that case we have $\left(2^{n}+m\right)^{2^{n}+m} \equiv m^{m}+2^{n} \not \equiv m^{m} \bmod 2^{n+1}$ and we obtain the desired contradiction.

So we conclude that no number from the lesser group and greater group has the same residue class as any other number from these two groups. This completes the induction step, and by induction it therefore follows that the given statement is true for all positive integers $n$.

2. We first prove that $\triangle C M B \sim \triangle D B E$. Since $D$ and $E$ lie on opposite sides of $M B$, it holds that $\angle D B E=\angle D B M+\angle M B E=\angle D B M+\angle M D B=$ $\angle C M B$ because of the given similarity and the exterior angle theorem. Moreover, it holds that

$$
\frac{|D B|}{|B E|}=\frac{|M D|}{|M B|}=\frac{|C M|}{|M B|}
$$

because of the similarity defining $E$ and the fact that $M$ is the midpoint of $C D$. It now follows that $\triangle C M B \sim \triangle D B E$ (sas). In particular, it follows that $\angle B E D=\angle M B C$. Therefore $\angle A B D=\angle M B C$ if and only if $\angle A B D=\angle B E D$. By the inscribed angle theorem (tangent case), this holds if and only if $A B$ is tangent to the circumcircle of $\triangle B D E$. The circle through $B$ and $D$ tangent to $A B$ is unique, and has as centre the intersection of the perpendicular bisector of $B D$ and the line through $B$ perpendicular to $A B$. So $A B$ is tangent to the circumcircle of $\triangle B D E$ if and only if $E$ lies on $\Omega$.
3. Answer: the smallest possible value is $3 \sqrt[3]{36}$.

Using the AM-GM inequality, we find

$$
\begin{aligned}
& x y+\frac{1}{3 x}+\frac{1}{2 y} \geq 3 \sqrt[3]{x y \frac{1}{3 x} \frac{1}{2 y}}=3 \sqrt[3]{\frac{1}{6}} \\
& y z+\frac{3}{2 y}+\frac{3}{z} \geq 3 \sqrt[3]{y z \frac{3}{2 y} \frac{3}{z}}=3 \sqrt[3]{\frac{9}{2}} \\
& x z+\frac{2}{3 x}+\frac{2}{z} \geq 3 \sqrt[3]{x z \frac{2}{3 x} \frac{2}{z}}=3 \sqrt[3]{\frac{4}{3}}
\end{aligned}
$$

When we add these three inequalities, we get

$$
\begin{aligned}
x y+y z+z x+\frac{1}{x}+\frac{2}{y}+\frac{5}{z} & \geq 3\left(\sqrt[3]{\frac{1}{6}}+\sqrt[3]{\frac{9}{2}}+\sqrt[3]{\frac{4}{3}}\right) \\
& =3\left(\frac{1}{6}+\frac{1}{2}+\frac{1}{3}\right) \sqrt[3]{36}=3 \sqrt[3]{36}
\end{aligned}
$$

For each of the three inequalities, equality holds if and only if the three terms on the left-hand side are equal. Solving the resulting system of equations, we get the solution $(x, y, z)=\left(\frac{1}{3} \sqrt[3]{6}, \frac{1}{2} \sqrt[3]{6}, \sqrt[3]{6}\right)$. And in that case $x y+y z+z x+\frac{1}{x}+\frac{2}{y}+\frac{5}{z}$ is equal to

$$
\begin{aligned}
\frac{1}{6} \sqrt[3]{36}+\frac{1}{2} \sqrt[3]{36}+\frac{1}{3} \sqrt[3]{36}+3 \frac{1}{\sqrt[3]{6}}+4 \frac{1}{\sqrt[3]{6}}+5 \frac{1}{\sqrt[3]{6}}=\sqrt[3]{36}+12 \frac{1}{\sqrt[3]{6}} \\
=\sqrt[3]{36}+2 \sqrt[3]{36}=3 \sqrt[3]{36}
\end{aligned}
$$

4. Answer: $M=3 n-4$ and $f\left(a_{1}, a_{2}, \ldots, a_{n}\right)=3 n-4$ if and only if $a_{1}=$ $a_{2}=\cdots=a_{n}=3$.
First of all, we note that for every distribution, there exists a move such that $\max \left(a_{1}, \ldots, a_{n}\right)$ increases by at least 1 , unless all the stones are in a single box. To see this, pick a box containing the highest number of stones, pick a different non-empty box and distribute the stones from that box such that at least one stone goes into the box containing the highest number of stones. It follows that $f\left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq 3 n-\max \left(a_{1}, \ldots, a_{n}\right)$. Specifically, if $\max \left(a_{1}, \ldots, a_{n}\right) \geq 5$, then $f\left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq 3 n-5$. The rest of the proof will follow from the following four claims.
Claim 1. If $\max \left(a_{1}, \ldots, a_{n}\right)=4$, then $f\left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq 3 n-5$.
Proof. Let $A_{1}$ be the box containing the highest number of stones, and $A_{2}$ be the box containing the second-highest number of stones. Note that we must have $a_{1}=4$ and $a_{2} \geq \frac{3 n-4}{n-1}=3-\frac{1}{n-1}$. Since $a_{2}$ is integer and $n \geq 3$, that means $a_{2} \geq 3$. While there exists a box other than $A_{1}$ or $A_{2}$ containing 2 or more stones, do the move consisting of distributing the stones in that box in such a way that $A_{1}$ and $A_{2}$ each receive a stone. The distribution $b_{1}, \ldots, b_{n}$ we obtain by performing these moves has the properties that $b_{3}+\cdots+b_{n} \leq n-2$ and that $b_{1}-b_{2}=a_{1}-a_{2} \leq 1$. Therefore we have $b_{1}+b_{2} \geq 3 n-(n-2)=2 n+2$, from which it follows that that $b_{2}=\frac{1}{2}\left(b_{2}+b_{2}\right) \geq \frac{1}{2}\left(b_{1}+b_{2}-1\right) \geq \frac{1}{2}(2 n+1)=n+\frac{1}{2}$. Since $b_{2}$ is integer, we therefore have $b_{2} \geq n+1$. Now we can do a move consisting of distributing $A_{2}$ 's stones in such a way that $A_{1}$ receives 2 stones. After that, while there exists a box other than $A_{1}$ that contains stones, we do a move consisting of distributing the stones in that box in such a way that $A_{1}$ receives 1 stone. Since $A_{1}$ receives at least two stones during one of
the moves, and at least one during each of the other moves, the number of moves needed to make all boxes except $A_{1}$ empty is at most $3 n-5$.
Claim 2. If $\max \left(a_{1}, \ldots, a_{n}\right)=3$, then $f\left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq 3 n-4$.
Proof. We make a random move and apply Claim 1 to the result. Then we are done in at most $1+(3 n-5)=3 n-4$ moves.
Claim 3. There are no moves so that the maximum $\max \left(a_{1}, \ldots, a_{n}\right)$ increases by 3 or more.
Proof. We proceed by contradiction. To make a move in which a box receives more than 3 stones, the box that was distributed from in that move would have to contain at least $3 n+1$ stones. This contradicts the fact that there are only $3 n$ stones. To make a move in which a box receives exactly 3 stones, the box that was distributed from should contain at least $2 n+1$ stones. Any box that contains the highest number of stones after this move, must contain at least $2 n+4$ stones, as the maximum has increased by at least 3 . That box therefore must also have contained at least $2 n+1$ stones before this move. This requires $4 n+2$ stones and is therefore a contradiction.
Claim 4. If $\max \left(a_{1}, \ldots, a_{n}\right)=3$, then $f\left(a_{1}, a_{2}, \ldots, a_{n}\right) \geq 3 n-4$.
Proof. Suppose for a contradiction that we can move all the stones into a single box in $3 n-5$ or fewer moves. Because of Claim 3, there are no moves where the maximum increases by 3 or more. That means there are at least two moves where the maximum increases by 2 . Such moves we will call large moves. Note that each box receives at least 1 stone on a big move. Let move $i$ be the first large move, and let move $j$ be the last large move. Since a large move can only be performed with a box containing at least $n+1$ stones and each box starts with 3 stones $i-1 \geq(n+1)-3$, or equivalently $i \geq n-1$.
Let $m$ be the number of empty boxes at the beginning of move $j$. Since each box contains at least 1 stone after move $i$, we must have made at least one move per box that is empty by the beginning of move $j$. Therefore $(j-1)-i \geq m$. Since each box receives at least 1 stone again, after move $j$ there are at most $m$ boxes containing exactly 1 stone; the other boxes contain at least 2 stones each. Since we don't make any more large moves after move $j$, it therefore takes at least $m+2(n-1-m)=2(n-1)-m$ moves afterwards to empty $n-1$ of the boxes. So the total number of moves is at least

$$
i+(j-i)+2(n-1)-m \geq(n-1)+(m+1)+2(n-1)-m=3 n-2,
$$

which contradicts our assumption that we could put all the stones into a single box in $3 n-5$ or fewer moves.

## IMO Team Selection Test 3, June 2023

Problems

1. Find all prime numbers $p$ for which the positive integer

$$
3^{p}+4^{p}+5^{p}+9^{p}-98
$$

has at most 6 positive divisors.
Remark. You are allowed to use the fact that 9049 is a prime number without proof.
2. Each pupil in the Netherlands is given a finite number of cards. On each card, there is a real number in the interval $[0,1]$. (The numbers on different cards do not have to be different.) Find the smallest real number $c>0$ for which the following holds, independent of the numbers on the cards each person has been given.

Any pupil for who the sum of the numbers on their cards is at most 1000, can distribute their cards over 100 boxes such that the sum of the cards in each box is at most $c$.
3. Let $\triangle A B C$ be an isosceles triangle with $|A B|=|A C|$. Given a point $P$ in $\triangle A B C$ distinct from to the circumcentre. Let $\omega$ be the circle through $C$ with centre $P$. The circle $\omega$ intersects the line segments $B C$ and $A C$ a second time in $D$ and $E$ respectively. Let $\Gamma$ be the circumcircle of $\triangle A E P$ and let $F$ be the second intersection point of $\omega$ and $\Gamma$. Prove that the centre of the circumcircle of $\triangle B D F$ lies on $\Gamma$.
4. Find all functions $f: \mathbb{Q}^{+} \rightarrow \mathbb{Q}$ such that

$$
f(x)+f(y)=\left(f(x+y)+\frac{1}{x+y}\right)(1-x y+f(x y))
$$

for all $x, y \in \mathbb{Q}^{+}$.

## Solutions

1. Write $f(p)=3^{p}+4^{p}+5^{p}+9^{p}-98$. We claim that the only prime numbers for which $f(p)$ has at most 6 positive divisors are 2,3 , and 5 .
Note that we have the prime factorisations $f(2)=3 \cdot 11, f(3)=7 \cdot 11^{2}$, and $f(5)=7 \cdot 9049$. Therefore $f(2), f(3), f(5)$ has $4,6,4$ positive divisors respectively, as the number of positive divisors of the prime factorisation $p_{1}^{e_{1}} p_{2}^{e_{2}} \cdots p_{n}^{e_{n}}$ is $\left(e_{1}+1\right)\left(e_{2}+1\right) \cdots\left(e_{n}+1\right)$.
Now let $p>5$, and suppose for a contradiction that $f(p)$ has at most 6 positive divisors. First note that modulo 7, we have that $f(p) \equiv 3^{p}+$ $(-3)^{p}+5^{p}+(-5)^{p}-0 \equiv 0$ as $p$ is odd.
Next consider $f(p)$ modulo 11. By Fermat's little theorem, we have $a^{10} \equiv 1$ $\bmod 11$ for any integer $a \not \equiv 0 \bmod 11$, so the residue class of $f(p)$ modulo 11 is constant on any residue class modulo 10 . Note that $p$ must be one of $1,3,-3,-1 \bmod 10$, since otherwise $p$ would have contained either 2 or 5 as a non-trivial factor.
Now note that by a straightforward computation $\left\{3^{3}, 4^{3}, 5^{3}, 9^{3}\right\}$ contains the same residue classes modulo 11 as $\{3,4,5,9\}$. Repeating the same argument shows that the same holds for $\left\{3^{-1}, 4^{-1}, 5^{-1}, 9^{-1}\right\}\left(\right.$ as $\left.-1 \equiv 3^{2} \bmod 10\right)$ and $\left\{3^{-3}, 4^{-3}, 5^{-3}, 9^{-3}\right\}$ as well. As $f(p) \equiv 3^{p}+4^{p}+5^{p}+9^{p}-98 \bmod 11$, it follows $f(p)$ has the same value modulo 11 for every $p \equiv 1,3,-3,-1$ $\bmod 10$, and therefore for every $p>5$.
We use the case $p \equiv 1 \bmod 10$ to compute this value:

$$
\begin{aligned}
f(p) & \equiv 3^{1}+4^{1}+5^{1}+9^{1}-98 \quad \bmod 11 \\
& \equiv 3+4+5+9+1 \quad \bmod 11 \\
& \equiv 0 \quad \bmod 11
\end{aligned}
$$

We deduce that $11 \mid f(p)$ for all $p>5$.
Now note that for all $p>5$ we have $f(p)>9^{5}=9 \cdot 81 \cdot 81>7 \cdot 11 \cdot 77$. Hence we can write $f(p)=7 \cdot 11 \cdot d$ with $d>77$. Then we see that $f(p)$ has at least 8 positive divisors, namely

$$
1<7<11<77<d<7 d<11 d<77 d .
$$

So $p=2,3,5$ are indeed the only prime numbers for which $f(p)$ has at most 6 positive divisors.
2. Suppose one of the pupils has been given 1001 cards, each containing the number $\frac{1000}{1001}$. Since the sum of the cards is 1000 , this pupil should be able to distribute the cards among the 100 boxes. Because of the pigeonhole principle, there is at least one box with 11 cards. The sum of these 11 cards is $11 \cdot \frac{1000}{1001}=11\left(1-\frac{1}{1001}\right)=11-\frac{11}{1001}=11-\frac{1}{91}$. We are now going to show that this is the smallest possible value, i.e. $c=11-\frac{1}{91}$.
For a random pupil, we first consider those distributions for which the maximum of the sums of the cards per box is as small as possible. From these distributions we then pick a distribution for which the number of boxes having their sum equal to this maximum, is as small as possible. Let $d_{1} \leq d_{2} \leq \ldots \leq d_{100}$ be the sums corresponding to the 100 boxes in this distribution, ordered from low to high (with the last $k$ of them equal to the maximum). Since the sum of all cards is at most 1000 , we have that

$$
99 d_{1}+d_{100} \leq d_{1}+d_{2}+\ldots+d_{100} \leq 1000 .
$$

On the other hand, moving a positive card from the box (with sum) $d_{100}$ to the box (with sum) $d_{1}$ cannot create a better distribution per assumption: i.e. the new distribution does not have a smaller maximum, or less than $k$ boxes equal to this maximum value. This means that the new value of $d_{1}$ is at least equal to $d_{100}$. If $d_{100} \leq 10$ we are immediately done, because $10<11-\frac{1}{91}$. So we may assume that $d_{100}>10$. Since each card is at most 1 , this implies that box $d_{100}$ contains at least 11 positive cards. This in turn implies that there is a card in this box with positive value at most $\frac{d_{100}}{11}$. Therefore, if we move this card to box $d_{1}$, then it must hold that

$$
d_{1}+\frac{d_{100}}{11} \geq \text { "new value of box } d_{1} " \geq d_{100} .
$$

We can rewrite this as $11 d_{1} \geq 10 d_{100}$. Combining this with the first equation, we find

$$
91 d_{100}=90 d_{100}+d_{100} \leq 99 d_{1}+d_{100} \leq 1000 .
$$

So, for each pupil, the smallest maximum of the sums of the cards per box is $d_{100} \leq \frac{1000}{91}=\frac{1001}{91}-\frac{1}{91}=11-\frac{1}{91}$.

3. Since $|F P|=|P E|$, the angles on these chords of $\Gamma$ are also equal: $\angle F A P=$ $\angle P A E=\angle P A C$. Then, using the cyclic quadrilateral $A E P F$, we find that $\angle P F A=180^{\circ}-\angle P E A=\angle P E C$. It follows that

$$
\begin{aligned}
\angle A P F & =180^{\circ}-\angle P F A-\angle F A P=180^{\circ}-\angle P E C-\angle P A C \\
& =180^{\circ}-\angle E C P-\angle P A C=\angle C P A .
\end{aligned}
$$

Since we also know that $|F P|=|P C|$ (and of course $|A P|=|A P|$ ) it follows that $\triangle A F P \simeq \triangle A C P$. From this we conclude that $|A F|=|A C|=|A B|$. This means that $A$ is the circumcentre of $\triangle B C F$. It also means that $\triangle A B F$ is isosceles and so the perpendicular bisector of $B F$ is equal to the bisector of $\angle B A F$. On the other hand, $\triangle D P F$ is also isosceles, so the perpendicular bisector of $D F$ is equal to the bisector of $\angle D P F$. We let $Q$ be the intersection of the perpendicular bisectors of $B F$ and $D F$. Since $A$ and $P$ are the centres of the corresponding circumscribed circles, we get with the centre-to-center angle theorem that

$$
\angle Q A F=\frac{1}{2} \angle B A F=\angle B C F=\angle D C F=\frac{1}{2} \angle D P F=\angle Q P F .
$$

So $Q$ lies on $\Gamma$.
4. Answer: the only function that satisfies the conditions is $f\left(\frac{m}{n}\right)=\frac{m}{n}-\frac{n}{m}$. Indeed, this function satisfies the conditions because

$$
\begin{aligned}
\left(f(x+y)+\frac{1}{x+y}\right) & (1-x y+f(x y)) \\
& =\left((x+y)-\frac{1}{x+y}+\frac{1}{x+y}\right)\left(1-x y+x y-\frac{1}{x y}\right) \\
& =(x+y)\left(1-\frac{1}{x y}\right)=x+y-\frac{1}{x}-\frac{1}{y}=f(x)+f(y) .
\end{aligned}
$$

For functions on $\mathbb{Q}^{+}$, it is often a good idea to see what happens to the natural numbers first. The idea is to play out products $x y$ against sums $x+y$, such as $2 \cdot 2=2+2$.
Lemma. We must have $f(1)=0$.
Proof. If we substitute $x=y=1$ we get $2 f(1)=\left(f(2)+\frac{1}{2}\right) f(1)$. This means that $f(1)=0$ or $f(2)=\frac{3}{2}$.
Suppose $f(2)=\frac{3}{2}$. If we now substitute $x=y=2$, we get $2 f(2)=$ $\left(f(4)+\frac{1}{4}\right)(f(4)-3)$. We expand this as $3=f(4)^{2}-\frac{11}{4} f(4)-\frac{3}{4}$. After multiplying by 4 , we see that we can decompose this as $(f(4)+1)(4 f(4)-$ $15)=0$. So $f(4)=-1$ or $f(4)=\frac{15}{4}$.
If we substitute $y=1$ in the original equation we find that

$$
\begin{equation*}
f(x)+f(1)=\left(f(x+1)+\frac{1}{x+1}\right)(f(x)-x+1) . \tag{8}
\end{equation*}
$$

If we substitute $x=2,3,4,5$ here, we get, respectively

$$
\begin{aligned}
\frac{3}{2}+f(1) & =\left(f(3)+\frac{1}{3}\right) \cdot \frac{1}{2}, \\
f(3)+f(1) & =\left(f(4)+\frac{1}{4}\right)(f(3)-2), \\
f(4)+f(1) & =\left(f(5)+\frac{1}{5}\right)(f(4)-3), \\
f(5)+f(1) & =\left(f(6)+\frac{1}{6}\right)(f(5)-4) .
\end{aligned}
$$

In the case that $f(4)=\frac{15}{4}$, the second equation yields that $f(3)+f(1)=$ $4(f(3)-2)$ so $3 f(3)=f(1)+8$. However, the first equation yields that $3 f(3)+1=3 \cdot 2 \cdot\left(\frac{3}{2}+f(1)\right)=6 f(1)+9$. We conclude that $f(1)+8=$ $3 f(3)=6 f(1)+9-1=6 f(1)+8$. We find that $f(1)=0$, as we wanted to prove.

On the other hand, suppose that $f(4)=-1$. As we have just seen, the first equation yields that $3 f(3)-6 f(1)=8$. In this case, the second equation yields that $f(3)+f(1)=-\frac{3}{4}(f(3)-2)$ so $7 f(3)+4 f(1)=6$. If we solve this system (e.g. by adding two times the first equation to three times the second), we find that $f(1)=-\frac{19}{27}$ and $f(3)=\frac{34}{27}$. If we substitute this into the third equation, we get $-1-\frac{19}{27}=-4\left(f(5)+\frac{1}{5}\right)$, or $f(5)=\frac{61}{270}$. Similarly, from the fourth equation we then get that $\frac{61}{270}-\frac{19}{27}=\left(f(6)+\frac{1}{6}\right)\left(\frac{61}{270}-4\right)$, or $f(6)=-\frac{245}{6114}$.
We have now calculated 6 as $4+1+1$, but of course we can also do it as $2 \cdot 3$. If we substitute $x=2$ and $y=3$, we get

$$
f(2)+f(3)=\left(f(5)+\frac{1}{5}\right)(f(6)-5) .
$$

However, if we substitute the values of $f(2), f(3), f(5)$ and $f(6)$ we found, we do not get equality, as the left side is greater than zero and the right side is smaller than zero. We conclude that the case $f(4)=-1$ cannot occur, and hence that $f(1)=0$.
Now we would like to use equation (8) inductively, but the base case $x=1$ does not work, because $f(1)=0$. As a new base case we would like to use $f(4)=\frac{15}{4}$ from before, which did not result in a contradiction, but rather in $f(1)=0$.
Lemma. We must have $f(4)=\frac{15}{4}$.
Proof. We actually use the same trick as above to calculate $f(4)$, but now with the fact that we know $f(1)$ instead of $f(2)$. With $x=y=2$, we again find that $2 f(2)=\left(f(4)+\frac{1}{4}\right)(f(4)-3)$. By substituting $x=2$ in (8) we find $f(2)=\left(f(3)+\frac{1}{3}\right)(f(2)-1)$ and by substituting $x=3$ in (8) we find $f(3)=\left(f(4)+\frac{1}{4}\right)(f(3)-2)$. We can solve this system of three equations by rewriting the second equation as

$$
f(3)=\frac{f(2)}{f(2)-1}-\frac{1}{3}=\frac{1}{f(2)-1}+\frac{2}{3},
$$

or $f(2)=\frac{1}{f(3)-\frac{2}{3}}+1$. Similarly, it follows from the third equation that $f(4)=\frac{2}{f(3)-2}+\frac{3}{4}$ which we can rewrite as $f(3)-\frac{2}{3}=\frac{\frac{4}{3} f(4)+1}{f(4)-\frac{3}{4}}$. Combining everything, we find that

$$
\begin{aligned}
\left(f(4)+\frac{1}{4}\right)(f(4)-3) & =2 f(2)=2\left(\frac{1}{f(3)-\frac{2}{3}}+1\right)=2\left(\frac{f(4)-\frac{3}{4}}{\frac{4}{3} f(4)+1}+1\right) \\
& =\frac{2 f(4)-\frac{3}{2}+\frac{8}{3} f(4)+2}{\frac{4}{3} f(4)+1}=\frac{\frac{14}{3} f(4)+\frac{1}{2}}{\frac{4}{3} f(4)+1}
\end{aligned}
$$

If we expand this by multiplying by $12\left(\frac{4}{3} f(4)+1\right)$ we get a cubic equation. With $t=f(4)$, this equation is $16 t^{3}-32 t^{2}-101 t-15=0$. Fortunately, we already have a conjecture for a zero, namely $t=\frac{15}{4}$. We can easily check that this is indeed a zero and we factor

$$
0=16 t^{3}-32 t^{2}-101 t-15=(4 t-15)\left(4 t^{2}+7 t+1\right)
$$

The discriminant of $4 t^{2}+7 t+1$ is $D=7^{2}-4 \cdot 4=33$ which is not a square of a rational number. So $t=\frac{15}{4}$ is the only rational zero of the cubic equation and the only possibility for $f(4)$.
From the three equations we started the system with for $f(4)$, it now follows directly that $f(2)=\frac{3}{2}$ and $f(3)=\frac{8}{3}$. Altogether, with $f(1), f(2), f(3)$ and $f(4)$, we have an induction basis for the statement $f(n)=n-\frac{1}{n}$ for all natural numbers $n$. Assume as an induction hypothesis that this formula holds for $n$. Then it follows from (8) that

$$
\begin{aligned}
f(n+1) & =\frac{f(n)+f(1)}{f(n)-n+1}-\frac{1}{n+1}=\frac{n-\frac{1}{n}+0}{n-\frac{1}{n}-n+1}-\frac{1}{n+1} \\
& =\frac{n^{2}-1}{n-1}-\frac{1}{n+1}=(n+1)-\frac{1}{n+1}
\end{aligned}
$$

which completes the induction. If we now substitute $x=n$ and $y=\frac{1}{n}$, we find

$$
f(n)+f\left(\frac{1}{n}\right)=\left(f\left(n+\frac{1}{n}\right)+\frac{1}{n+\frac{1}{n}}\right)(1-1+0)=0
$$

It follows that $f\left(\frac{1}{n}\right)=-f(n)=\frac{1}{n}-n$. Now we prove that $f\left(\frac{m}{n}\right)=\frac{m}{n}-\frac{n}{m}$ with induction to $m$. We just proved the induction basis with $m=1$. So now suppose this formula holds for some $m$ and all $n$. Now we substitute $x=\frac{m}{n}$ and $y=\frac{1}{n}$ from which we get that

$$
\begin{aligned}
f\left(\frac{m+1}{n}\right) & =\frac{f\left(\frac{m}{n}\right)+f\left(\frac{1}{n}\right)}{f\left(\frac{m}{n^{2}}\right)-\frac{m}{n^{2}}+1}-\frac{n}{m+1}=\frac{\frac{m}{n}-\frac{n}{m}+\frac{1}{n}-n}{\frac{m}{n^{2}}-\frac{n^{2}}{m}-\frac{m}{n^{2}}+1}-\frac{n}{m+1} \\
& =\frac{(m+1)\left(\frac{1}{n}-\frac{n}{m}\right)}{n\left(\frac{1}{n}-\frac{n}{m}\right)}-\frac{n}{m+1}=\frac{m+1}{n}-\frac{n}{m+1}
\end{aligned}
$$

This completes the induction to $m$, and with it the complete proof.

## Junior Mathematical Olympiad, September 2022

## Problems

## Part 1

1. Joah has a very long liquorice lace. He keeps taking bites out of the lace (but not from the very beginning or end of the lace), each time eating 2 cm of the liquorice, creating two smaller pieces in the process. He repeats this several times. At the end, he is left with pieces of liquorice lace of $1,2,3$, $4,5,6,7,8,9$, and 10 cm .
How long (in cm ) was his liquorice lace originally?
A) 55
B) 66
C) 73
D) 75
E) 81
2. Five distinct positive integers are in a sequence ordered from small to large. The middle number is 20 . The difference between the smallest two numbers equals the difference between the largest two numbers. The fourth number is four times as large as the first number, and the fifth number is twice as large as the second number.
When you add all five numbers, what is the outcome?
A) 84
B) 90
C) 104
D) 110
E) 130
3. Petra, Quinten, Rakhi, Salome, and Teun organise a badminton tournament consisting of five rounds. In each round, two players play against each other and a third player is the referee. The other two players rest during the round. Everyone plays twice and is the referee once. Nobody plays two matches in a row and the referee of a match always rests in the next round. Salome and Teun face each other in the first round. In the third round, Rakhi plays against Salome, while Quinten is resting. Who is the referee of the fifth round?
A) Petra
B) Quinten
C) Rakhi
D) Salome
E) Teun
4. The sides of a triangle have lengths $13, x$, and $2 x$. Here $x$ is an integer. How many possibilities are there for $x$ ?
A) 2
B) 6
C) 7
D) 8
E) 12
5. On a long street, there are four houses, numbered from 1 to 4 , where the distances between the houses are all distinct. The houses have their front door directly on the street. There are eight people living in the first house, two people each in the second and the third house, and three people in the fourth house. A new bus stop is constructed in the street, in such a way that the total distance for the 15 inhabitants of the street to the bus stop is as short as possible.
Which house will be closest to the bus stop?
A) House 1
B) House 2
C) House 3
D) House 4
E) That depends on the distances between the houses.
6. Ayman writes down the numbers 1 through 10 in a sequence in some order, writes down the nine (positive) differences between adjacent numbers and computes the sum of these differences. The result is called the dynamic of the sequence. For example, the dynamic of the sequence $1,2,3,4,5,6,7,8,9,10$ is 9 , and the dynamic of $2,1,3,10,4,5,9,6,8,7$ is $1+2+7+6+1+4+3+2+1=27$.
What is the greatest dynamic that such a sequence with the numbers 1 through 10 can have?
A) 41
B) 43
C) 45
D) 47
E) 49
7. There are 25 guests at a party, one of which is Medan. Among the other guests, there are 12 that each shook hands with exactly 18 people. The other 12 each shook hands with exactly 6 people.
With how many guests did Medan shake hands?
A) 0
B) 6
C) 12
D) 18
E) 24
8. Sil has a lot of cards, which are yellow on one side and blue on the other. Most cards have a number on both sides. If two cards have the same number on the yellow side, then they have the same number on the blue side. There are also cards with $\mathrm{a} \times$ on the yellow side and $\mathrm{a}+$ on the blue side. Finally, there are cards which have an $=$ sign on both sides. If you put down a correct multiplication with some of the yellow cards and then turn over these cards, then you get a correct addition in blue. Cards with a 2 on the yellow side have a 2 on the blue side, cards with a 3 on the yellow
side have a 3 on the blue side, and cards with a 5 on the yellow side have a 5 on the blue side.

All cards are lying on the table with the yellow side facing up. Sil tries to discover what is on the blue side, without turning over the cards. For example, cards with a 6 on the yellow side have a 5 on the blue side, because the yellow expression $2 \times 3=6$ must have $2+3=5$ on the blue back. Cards with a 20 on the yellow side have a 9 on the blue back side, because the yellow expression $2 \times 2 \times 5=20$ becomes $2+2+5=9$ in blue. The back of a yellow card containing a fraction, for example $\frac{5}{3}$, can be determined using $\frac{5}{3} \times 3=5$, which becomes $2+3=5$ when flipped; hence on the blue side is a 2 .

For which of the following numbers on the yellow side will there be a negative number on the blue side?
A) $\frac{9}{8}$
B) $\frac{25}{27}$
C) $\frac{32}{27}$
D) $\frac{64}{81}$
E) $\frac{128}{125}$

## Part 2

1. Eleonora has a piece of paper in the shape of an equilateral triangle with an area of 1 . She folds the piece several times and puts it flat on the table. It turns out that the figure on the table is not more than four layers thick anywhere.
What is the minimum area of the figure lying on the table?
2. A zoo is reconstructing part of their park. In this part, there will be six areas with six species of animals, one in each area. The six species are tigers, lions, elephants, giraffes, zebras and monkeys. The map is as follows:


The tigers and lions cannot be next to each other (this means not in two areas which share a side as border; two areas bordering in a vertex are allowed). The monkeys cannot be next to the tigers and also not next to the lions. The zebras cannot be next to the tigers.
In how many ways can the zoo distribute the six species over the six areas?
3. We draw a rectangle in a grid. The four midpoints of the four sides of the rectangle turn out to be the vertices $(-3,0),(2,0),(5,4)$, and $(0,4)$.
What is the area of the rectangle?
4. A whiteboard contains a calculation $1 ? 2 ? 3 ? 4 ? 5 ? 6$, where each question mark is either a + or $\mathrm{a} \times$. The correct outcome of the calculation is written on the back of the board. Jaap copies the calculation but accidentally turns one of the plus signs into a times sign. The outcome is now 58 more than the number on the back of the board. Jaap now changes a times sign back into a plus sign, but not on the place where he made the mistake before. Now the result differs 1 from the previous result.
What number is on the back of the board?
5. We construct a sequence of numbers starting with 2022 and 21. Each next number in the sequence is equal to the positive difference of the two previous numbers. So the third and fourth number in the sequence are 2001 and 1980.
At which place in the sequence do we find the number 0 for the first time?
6. Kjell has a large piece of graph paper of $100 \times 100$ squares.

How many squares can Kjell colour at most without there being three coloured squares in a row, all directly next to each other or all directly above each other?
7. In an apartment building with floors 0 up to and including 10 , there is one person living on each floor. Each morning, everyone in the building must go to floor 0 to go outside. Everyone is willing to walk the stairs for at most three floors. There can be at most four people in the lift at the same time. The lift starts at floor 0 .
At least how many floors must the lift move to get everyone outside? Movements up and movements down are both counted.
8. A triple of consecutive two-digit positive integers is called sweet if the four-digit number formed by the first and the last number is divisible by the middle number. For example, the triple $(20,21,22)$ is not sweet, because 2022 is not divisible by 21 . Note that a two- or four-digit number cannot start with a 0 , so 03 is not a two-digit number, for example.
How many sweet triples are there?

## Answers

## Part 1

1. C) 73
2. A) Huis 1
3. C) 104
4. E) 49
5. 

B) Quinten
7. C) 12
4. D) 8
8. E) $\frac{128}{125}$

Part 2

1. $\frac{1}{4}$
2. 154
3. 16
4. 6667
5. 40
6. 14
7. 68
8. 2

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