We eat problems for breakfast.

Preferably unsolved ones...

48th Dutch Mathematical Olympiad 2009

NEDERLANDSE WISKUNDE OLYMPIADE

Contents

Introduction

In 2009 the Dutch Mathematical Olympiad consisted of two rounds. The first round was held on 30 January 2009 at the participating schools. The paper consisted of eight multiple choice questions and four open-answer questions, and students got two hours to work on it. In total 4379 students of 230 secondary schools participated in this first round.

The best students were invited for the final round. In order to stimulate young students to participate, we set different thresholds for students from different grades. Those students from grade 5 (4, ≤ 3) that scored 26 (23, 20) points or more on the first round (out of a maximum of 36 points) were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited.

For the second consecutive year we organised training sessions at six universities in the country for the 155 students who had been invited for the final round. Former Dutch IMO-participants were involved in the training sessions at each of the universities.

Out of those 155, in total 131 participated in the final round on 18 September 2009 at Eindhoven University of Technology. This final round contained five problems for which the students had to give extensive solutions and proofs. They had three hours for the paper. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 48th edition 2009. In 2011 we will have our 50th edition.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad 2009 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June.

On 5 March 2010 the first selection test was held. The best ten students participated in the second Benelux Mathematical Olympiad (BxMO), held in Amsterdam, the Netherlands.

In June, out of those 10 students and 3 reserve candidates, the team for the International Mathematical Olympiad 2010 was selected by two team selection tests on 9 and 12 June 2010. A seventh, young, promising student was selected to accompany the team to the IMO. The team had a training camp in Astana from 28 June until 5 July, together with the team from

New Zealand.

For younger students we organised the second Junior Mathematical Olympiad in October 2009 at the VU University Amsterdam. The students invited to participate in this event were the 30 best students of grade 1, grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with 15 multiple choice questions and one with 10 open-answer questions. The goal of this Junior Mathematical Olympiad is to scout talent and stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

The Dutch team for IMO 2010 Kazakhstan consists of

- Guus Berkelmans (16 y.o.)
- Harm Campmans (18 y.o., participated in IMO 2009 as well)
- Madelon de Kemp (17 y.o.)
- David Kok (17 y.o., participated in IMO 2009 as well)
- Daniël Kroes (16 y.o.)
- Merlijn Staps (15 y.o., observer C at IMO 2009)

We bring as observer C the promising young student

- Jeroen Huijben (14 y.o.)

The team is coached by

- Birgit van Dalen (deputy leader), Leiden University
- Johan Konter (team leader), Utrecht University
- Quintijn Puite (observer A), Eindhoven University of Technology
- Sietske Tacoma (observer B), Utrecht University

The Dutch delegation for IMO 2010 Kazakhstan further consists of

- Wim Berkelmans (member AB, observer A), VU University Amsterdam
- Karst Koymans (observer A), University of Amsterdam
- Aad Loois (observer C)
- Jelle Loois (observer A), ORTEC
- Ronald van Luijk (observer A), Leiden University
- Rozemarijn Schalkx (observer A), Eindhoven University of Technology

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

First Round, January 2009

Problems

A-problems

A1. Ella does three tests. During the first one, she answered 60% of the 25 questions correctly, during the second one, she answered 70% of the 30 questions correctly, and during the last one, she answered 80% of the 45 questions correctly. Now if we merge these three tests together to form one of 100 questions, what is the percentage of these 100 questions that Ella answered correctly?

A) 68% B) 70% C) 72% D) 74% E) 76%

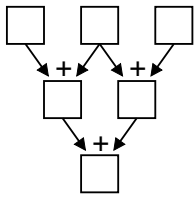
A2. How many of the integers from 10 to 99 (10 and 99 included) have the property that the sum of its digits is equal to the square of an integer? (An example: The sum of the digits of 27 is equal to $2+7=9=3^2$.)

A) 13 B) 14 C) 15 D) 16 E) 17

A3. Ronald throws three dice. These dice look just like ordinary dice, but their faces are numbered differently. The first die has the numbers 1, 1, 2, 2, 3 and 3 on it. The second die has the numbers 2, 2, 4, 4, 6 and 6 on it. And the third die has the numbers 1, 1, 3, 3, 5 and 5 on it. He then adds up the three numbers he gets from rolling the three dice. What is the probability that the resulting number is odd?

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

A4. Three distinct numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ are placed in the three squares in the top of the figure to the right, after which the numbers are added as described in said figure. We call Max the highest number that can appear in the bottom square, and Min the lowest number that can appear there. What is the value of Max − Min?



A) 16 B) 24 C) 25 D) 26 E) 32

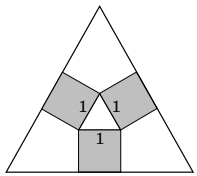
A5. The lengths of the diagonals of a rhombus have a ratio of $3:4$. (A rhombus is an equilateral quadrilateral.) The sum of the lengths of the diagonals is 56. What is the diameter of this rhombus?

A) 80 B) 96 C) 100 D) 108 E) 160

A6. Wouter is traveling by foot from his home to the fitness center. He also could have chosen to travel by bike, in which case he would travel 7 times as fast. But he left his bike at home. After walking for 1 km, continuing to walk would take just as long as walking back to get his bike, and then travel further by bike. By then, what is the distance in km to the fitness center?

A) $\frac{8}{7}$ B) $\frac{7}{6}$ C) $\frac{6}{5}$ D) $\frac{5}{4}$ E) $\frac{4}{3}$

A7. On the sides of an equilateral triangle, we draw three squares. The sides of these squares that are parallel to the sides of the triangle are extended until they intersect. These three intersections form another equilateral triangle. Suppose that the length of a side of the original triangle is equal to 1. What is the length of a side of the large equilateral triangle?



A) $1+2\sqrt{2}$ B) $5-\frac{1}{2}\sqrt{3}$ C) $3\sqrt{2}$
D) $1+2\sqrt{3}$ E) $2\sqrt{6}$

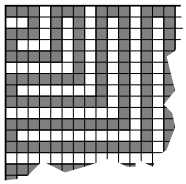
A8. Consider all four-digit numbers where each of the digit 3, 4, 6 and 7 occurs exactly once. How many of these numbers are divisible by 44?

A) 2 B) 4 C) 6 D) 8 E) 12

B-problems

The answer to each B-problem is a number.

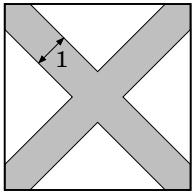
B1. On a sheet of paper, a grid of 101 by 101 white squares is drawn. A chain is formed by coloring squares grey as shown in the figure to the right. The chain starts in the upper left corner and goes on until it cannot go on any further. A large piece of the sheet is torn off. How many squares were colored grey in the original grid of 101 by 101 squares?



B2. The integer N consists of 2009 consecutive nines. A computer calculates $N^3 = (99999\ldots99999)^3$. How many nines does the number N^3 contain in total?

$$\underbrace{99999\ldots99999}_{2009\times}$$

B3. Using, a wide brush, we paint the diagonals of a square tile, as in the figure. Exactly half of the area of this tile is covered with paint. Knowing that as the width of the brush is 1, as indicated in the figure, what is the length of the side of the tile?



B4. Determine a triplet of integers (a, b, c) satisfying:

$$\begin{aligned} a + b + c &= 18 \\ a^2 + b^2 + c^2 &= 756 \\ a^2 &= bc \end{aligned}$$

Solutions

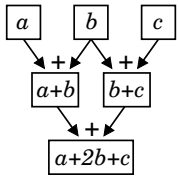
A-problems

A1. C) 72% 60% of 25 is 15; 70% of 30 is 21; and 80% of 45 is 36. So in total, Ella answered $15 + 21 + 36 = 72$ of the 100 questions correctly.

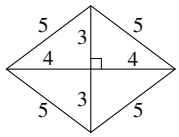
A2. E) 17 We check how many of these numbers have sum of digits equal to 1, 2, etc. There is 1 number with sum 1 (being 10); there are 2 with sum 2 (being 20 and 11); etc.; 9 with sum 9 (being $90, 81, \ldots, 18$); also, 9 with sum 10 (being $91, 82, \ldots, 19$); etc.; and finally, 1 with sum 18 (being 99); see the table below. Then the sum of digits is a square of an integer (i.e. 1, 4, 9 or 16) in $1+4+9+3 = 17$ of the 90 cases.

A3. B) $\frac{1}{3}$ Note that the second die only has even numbers on it, and that the third die only has odd numbers on it. So essentially the question is to find the probability that rolling the first die gives an *even* number. Since 2 of the 6 numbers on this die are even, this probability is equal to $\frac{2}{6} = \frac{1}{3}$.

A4. D) 26 Let's say we put a, b, and c in the top three squares. Then the result in the bottom square is $a+2b+c$. So we can maximize this result by making first b, then a and c as large as possible. Taking $b = 9, a = 8$ and $c = 7$ then yields 33 as result. In the same way, we can minimize the result by making first b, then a and c as small as possible. Taking $b = 1$, $a = 2$, $c = 3$ yields 7 as result. The difference between these numbers is $33 - 7 = 26$.

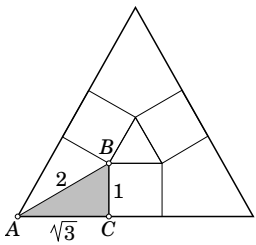


A5. A) 80 Note that the diagonals have lengths $\frac{3}{7} \cdot 56 = 24$ and $\frac{4}{7} \cdot 56 = 32$. So the halves of diagonals have lengths $12 = 3 \cdot 4$ and $16 = 4 \cdot 4$. So the rhombus is 4 times larger than the rhombus in the figure, which consists of four triangles with sides 3,4 and 5. Hence the sides of the original rhombus have length $4 \cdot 5 = 20$, and thus the diameter has length $4 \cdot 20 = 80$.



A6. E) $\frac{4}{3}$ Let x be said distance and let us suppose that he has spent a quarter of an hour walking by then. Then continuing walking will take him x quarters of an hour. On the other hand, if he decides to walk back home to pick up his bike, he'll first have to spend one quarter of an hour to get back, and then $\frac{1+x}{7}$ quarters of an hour by bike; since he travels 7 times faster that way. Then we have $x = 1 + \frac{1+x}{7}$, so $7x = 7 + (1 + x)$, or $6x = 8$. We deduce that $x = \frac{8}{6} = \frac{4}{3}$. Taking for 'quarter of an hour' any other time unit will give us the same result.

A7. D) $1 + 2\sqrt{3}$ In $\triangle ABC$, $\angle A$ is half of $60°$, so $30°$. Also, $\angle C$ is a right angle, so $\triangle ABC$ is a $30°$-$60°$-$90°$-triangle, where $|BC| = 1$. So it's half of an equilateral triangle with sides 2: $|AB| = 2$. Now we calculate $|AC|$ with the Theorem of Pythagoras: $|AC| = \sqrt{2^2 - 1^2} = \sqrt{3}$. So the required length is $\sqrt{3} + 1 + \sqrt{3}$.

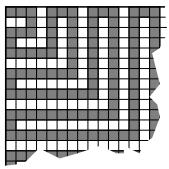


A8. A) 2 Suppose that n, having digits a, b, c and d (so $n = 1000a + 100b + 10c + d$) is divisible by 44. Then it is also divisible by 11. Since the number $m = 1001a + 99b + 11c$ is also divisible by 11, so is $m - n$. Hence $m - n = a - b + c - d$ is a multiple of 11. But this number is at most the sum of the two highest digits, minus the sum of the lowest two, so $13 - 7 = 6$, and in the same way, we see that this number is at least -6. So it has to be equal to 0. Thus $a + c = b + d$, and since the sum of the digits is 20, we have $a + c = b + d = 10$. First suppose that $d = 4$. Then $b = 6$ so we get two possibilities for n, namely 3674 and 7634. But neither of them is divisible by 4, let alone by 44. Now suppose that $d = 6$, then we have $b = 4$, and in this case we get 3476 and 7436, both of which are divisible by 44. Finally, note that since n is divisible by 4, d must be even. We deduce that we have only 2 such numbers that are divisible by 44.

Alternative solution Check all 24 possibilities, or just the 12 even possibilities, of even only the 6 multiples of 4.

B-problems

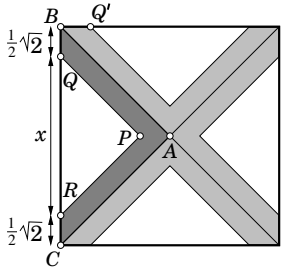
B1. 5201 We can subdivide this grid of 101^2 squares as follows. In the upper left corner, we have one (grey) square, then two L-shaped pieces, one having 3 squares (one of which grey), the other having 5 squares (all of which grey). Then we have two more L-shaped pieces, one having 7 squares (one of which grey), the other having 9 squares (all of which grey), etc. Of the last two L-shaped pieces, the first one has 199 squares (one of which grey), and the second one has 201 squares (all of which grey). We have 50 pairs of L-shapes in total, so the total number of grey squares is $1+(1+5)+(1+9)+(1+13)+\ldots+(1+201) = 1+(6+10+14+\ldots+202) = 1+\frac{1}{2}\cdot 50\cdot(6+202) = 5201$.



Alternative solution In each pair of these L-shapes, there are 4 more grey squares than white squares. So there are $50 \cdot 4 = 200$ more grey squares than white squares in the $101^2 - 1 = 10200$ squares contained in the 50 pairs of L-shapes, so we have 5000 white squares and 5200 grey ones. Since the upper left square is grey, in total, we have 5201 grey squares.

B2. 4017 $9^3 = 729$; $99^3 = 970299$; $999^3 = 997002999$. It seems to be the case that in general, the third power of a number n consisting of k consecutive nines takes the following form: first $k-1$ nines; then a 7; then $k-1$ zeroes; then a 2; and finally k nines. To prove this, we write $n = 10^k - 1$. Indeed: $(10^k-1)^3 = 10^{3k} - 3\cdot 10^{2k} + 3\cdot 10^k - 1 = 10^{2k}(10^k-3) + (3\cdot 10^k - 1)$. The number $10^k - 3$ can be written as $999\ldots997$ with $k-1$ nines. Multiplied with 10^{2k} this gives a number that ends in $2k$ zeroes. Adding $3\cdot 10^k - 1$ to this number, the last $k+1$ zeroes are replaced with $2999\ldots999$ with k nines. So in total, we have $(k-1)+k$ nines; in our case, $k = 2009$, so we have 4017 nines.

B3. $2+2\sqrt{2}$ We only need to look at a quarter of the tile: $\triangle ABC$. The area of $\triangle PQR$ is half of the area of $\triangle ABC$. The triangles are similar, so corresponding sides have a ratio of $1:\sqrt{2}$, so $|QR|:|BC| = 1:\sqrt{2}$. Now we calculate $|BQ|$ using the Theorem of Pythagoras in $\triangle BQQ'$: $2|BQ|^2 = |BQ'|^2 + |BQ|^2 = 1^2$, so $|BQ| = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$. Now let us write x for $|QR|$. Then we find $x+\sqrt{2} = \sqrt{2}\cdot x$, so $x(\sqrt{2}-1) = \sqrt{2}$ or equivalently, $x = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = 2+\sqrt{2}$. Hence $|BC| = x+\sqrt{2} = 2+2\sqrt{2}$ (or $|BC| = \sqrt{2}\cdot x = \sqrt{2}\cdot(2+\sqrt{2}) = 2\sqrt{2}+2$).



B4. $(a,b,c) = (-12,6,24)$ of $(a,b,c) = (-12,24,6)$ (one answer is enough)

We calculate $(b+c)^2$ in two different ways. $(b+c)^2 = (18-a)^2 = 324 - 36a + a^2$ and $(b+c)^2 = b^2 + 2bc + c^2 = (756 - a^2) + 2a^2$. So $a^2 - 36a + 324 = a^2 + 756$, or $-36a = 756 - 324 = 432$, so $a = -12$. Substituting this in the first and in the last equation, we obtain the equations $b + c = 30$ and $bc = 144$. Trying some divisors of $144 = 12^2$, we then should be able to find a solution. Or we can just substitute $c = 30 - b$ in the last equation, yielding the quadratic equation $b(30-b) = 144$, or equivalently $b^2 - 30b + 144 = 0$. We can factorize this as $(b-6)(b-24) = 0$ (or we can use the *abc*-formula) to see that we have two solutions $b = 6$ (and $c = 24$) or $b = 24$ (and $c = 6$).

Final Round, September 2009

Problems

For these problems not only the answer is important; you also have to describe the way you solved the problem.

1. In this problem, we consider integers consisting of 5 digits, of which the first and last one are nonzero. We say that such an integer is a *palindromic product* if it satisfies the following two conditions:

- the integer is a palindrome, (i.e. it doesn't matter if you read it from left to right, or the other way around);
- the integer is a product of two positive integers, of which the first, when read from left to right, is equal to the second, when read from right to left, like 4831 and 1384.

For example, 20502 is a palindromic product, since $102 \cdot 201 = 20502$, and 20502 itself is a palindrome.
Determine all palindromic products of 5 digits.

2. Consider the sequence of integers 0, 1, 2, 4, 6, 9, 12, ... obtained by starting with zero, adding 1, then adding 1 again, then adding 2, and adding 2 again, then adding 3, and adding 3 again, and so on. If we call the subsequent terms of this sequence a_0, a_1, a_2, ..., then we have $a_0 = 0$, and

$$a_{2n-1} = a_{2n-2} + n, \qquad a_{2n} = a_{2n-1} + n$$

for all integers $n \geqslant 1$.
Find all integers $k \geqslant 0$ for which a_k is the square of an integer.

3. A tennis tournament has at least three participants. Every participant plays exactly one match against every other participant. Moreover, every participant wins at least one of the matches he plays. (Draws do not occur in tennis matches.)
Show that there are three participants A, B and C for which the following holds: A wins against B, B wins against C, and C wins against A.

4. Let ABC be an arbitrary triangle. On the perpendicular bisector of AB, there is a point P inside of triangle ABC. On the sides BC and CA, triangles BQC and CRA are placed externally. These triangles satisfy $\triangle BPA \sim \triangle BQC \sim \triangle CRA$. (So Q and A lie on opposite sides of BC, and R and B lie on opposite sides of AC.)
Show that the points P, Q, C and R form a parallelogram.

5. We number a hundred blank cards on both sides with the numbers 1 to 100. The cards are then stacked in order, with the card with the number 1 on top.
The order of the cards is changed step by step as follows: at the 1st step the top card is turned around, and is put back on top of the stack (nothing changes, of course), at the 2nd step the topmost 2 cards are turned around, and put back on top of the stack, up to the 100th step, in which the entire stack of 100 cards is turned around. At the 101st step, again only the top card is turned around, at the 102nd step, the topmost 2 cards are turned around, and so on.
Show that after a finite number of steps, the cards return to their original positions.

Solutions

1. Note that the two integers mentioned in the second condition cannot end in a 0; since otherwise the product would also have to end in a 0. Thus, the two numbers have an equal number of digits. But a product of two integers of four or more digits consists of at least seven digits, hence is too large, and a product of two integers of at most two digits consists of at most four digits, hence is too small.
So to satisfy the second condition, we have to consider products of integers consisting of three digits, say $\overline{abc}$ and $\overline{cba}$.
If we work this out, we obtain: $\overline{abc} \cdot \overline{cba} = (100a + 10b + c)(100c + 10b + a) = 10000ac + 1000b(a+c) + 100(a^2+b^2+c^2) + 10b(a+c) + ac$.
If $ac > 9$, then this integer consists of more than 5 digits. Hence $ac \leqslant 9$. Now if $b(a+c) > 9$, then the first digit will be greater than ac, hence it will no longer be equal to the last digit. Hence we also have $b(a+c) \leqslant 9$. We can use a similar argument to show that $a^2+b^2+c^2 \leqslant 9$. We deduce from this that $1+b^2+1 \leqslant a^2+b^2+c^2 \leqslant 9$, and hence that b is equal to one of 0,1,2.

If $b = 0$, then $a^2 + c^2 \leqslant 9$, so we have $1 \leqslant a \leqslant 2$ and $1 \leqslant c \leqslant 2$.

If $b = 1$, the conditions become $a + c \leqslant 9$ and $a^2 + c^2 \leqslant 8$, and we must also have $1 \leqslant ac \leqslant 9$. From the second condition, we obtain $1 \leqslant a \leqslant 2$ and $1 \leqslant c \leqslant 2$.
If $b = 2$, the conditions become $a + c \leqslant 4$ and $a^2 + c^2 \leqslant 5$. Hence we have three possibilities for (a, b).
In the above table, we have worked out all the possibilities. It turns out that there are eight possible palindromic products: 10201, 12321, 14641, 20502, 23632, 26962, 40804 and 44944. $\square$

2. For $n \geqslant 1$, we have $a_{2n} = a_{2n-1} + n = a_{2n-2} + 2n$, so $a_{2n} = 0 + 2 + 4 + \cdots + 2n = 2(0 + 1 + 2 + \cdots + n) = n(n + 1)$. Since $n > 0$, we have $n^2 < n^2 + n < n^2 + 2n + 1 = (n+1)^2$. Hence $a_{2n} = n(n+1)$ cannot be a square; it lies between two subsequent squares. So for the even $k \geqslant 2$, a_k is not a square. From $a_{2n-1} = a_{2n} - n = n(n + 1) - n = n^2$, we deduce that a_{2n-1} is always a square, hence all odd k are as desired. Finally, we note that $a_0 = 0$ is a square as well. So a_k is a square if k is odd or $k = 0$, and it's not a square otherwise. $\square$

3. We first show that there is a cycle, i.e. m distinct participants $A_1, \ldots, A_m$ such that A_1 won against A_2, A_2 won against A_3, and so on, up to A_m, who won against A_1. Pick an arbitrary participant B_1. He won against another participant B_2, who won against a certain B_3, and so on. Since we have only finitely many players, we will have a repetition in our sequence of participants, i.e., B_1 won against B_2, B_2 won against B_3, $\ldots$, B_u won against B_{u+1}, where B_{u+1} is a participant that occurred earlier in our sequence, say B_t. Then B_t up to B_u indeed form a cycle. Note that the length m of a cycle is at least 3. In fact, we now need to show that there is a cycle of length 3.

Since there exists a cycle, we can take one of minimal length M, say $C_1, \ldots, C_M$. If $M > 3$, then we consider the match between C_1 and C_3. If C_3 won against C_1, then we have obtained a cycle of length 3, which is smaller than M, yielding a contradiction. On the other hand, if C_1 won against C_3, then, by removing C_2, we obtain a cycle of length $M - 1$, which is again smaller than M. This is a contradiction as well. We deduce that $M = 3$ and that C_1, C_2, C_3 is the desired cycle of length 3. $\square$

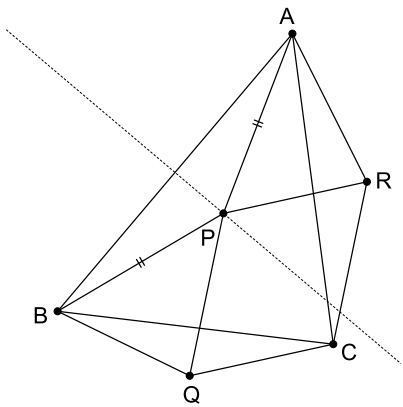
Alternative solution 1: Pick a participant A who has won the least number of matches, and a participant B who lost to A. Consider the set X of participants who lost to B. Then the set X does not contain A. If there is no triple of participants as desired, then A must have won against all participants in X. But A also won against B. But then A has won at least one match more than B, yielding a contradiction. $\square$

4. Note that $|AP| = |PB|$, and hence that $|BQ| = |QC|$ and $|CR| = |RA|$. Since $\triangle BPA \sim \triangle BQC$, it follows that $\frac{|PB|}{|AB|} = \frac{|QB|}{|CB|}$, and hence that $\frac{|PB|}{|QB|} = \frac{|AB|}{|CB|}$. Moreover, since Q lies outside triangle ABC and P lies inside of it, we have

$$\angle QBP = \angle QBC + \angle CBP = \angle PBA + \angle CBP = \angle CBA,$$

from which follows that $\triangle ABC \sim \triangle PBQ$. Note that $\triangle ABC$ is $\frac{|AB|}{|PB|}$ times larger than $\triangle PBQ$. Similarly, it follows that $\triangle ABC \sim \triangle APR$, and note that $\triangle ABC$ is $\frac{|AB|}{|PA|}$ times larger than $\triangle APR$. Since $|PA| = |PB|$, from which follows that $\frac{|AB|}{|PA|} = \frac{|AB|}{|PB|}$, we deduce that $\triangle PBQ \cong \triangle APR$. Hence $|QP| = |RA| = |CR|$ and $|PR| = |BQ| = |QC|$.

So the two pairs of opposite sides of quadrilateral $PQCR$ have equal length, hence quadrilateral $PQCR$ is a parallelogram. $\square$



5. We first consider the effect of two consecutive steps. If before step $2k-1$, the stack of cards, from top to bottom, consists of $a_1, \ldots, a_{100}$, the first $2k-1$ cards are reversed in order, so we obtain $a_{2k-1}, a_{2k-2} \ldots, a_2, a_1, a_{2k}, a_{2k+1} \ldots, a_{100}$. Then, at the $2k^{\text{th}}$ step, the top $2k$ cards are reversed in order, yielding $a_{2k}, a_1, a_2, \ldots, a_{2k-2}, a_{2k-1}, a_{2k+1}, \ldots, a_{100}$. Hence the effect of these two steps is that the $2k^{\text{th}}$ card is moved up to the top, and the rest is shifted down. So if we consider the first 100 steps, then first 2 is put on top, then 4 is put on top of that, then 6, and so on, so we obtain $100, 98, 96, \ldots, 4, 2, 1, 3, 5, \ldots, 97, 99$. We call this a megastep. In general, a megastep changes the order $a_1, a_2, \ldots, a_{99}, a_{100}$ into $a_{100}, a_{98}, a_{96}, \ldots, a_4, a_2, a_1, a_3, a_5, \ldots, a_{97}, a_{99}$. This megastep is invertible; the order after a megastep can only come from one unique order. Now we there are only finitely many possible ways (namely 100!) to order the 100 cards, so after applying megasteps for a while, we'll obtain an order that has occurred before. Consider the first megastep that yields an order that has occurred before. If this is not the starting position, then one megastep before, the position should have occurred earlier as well; a contradiction. Hence we return to the starting position after a finite number of (mega)steps. $\square$

Alternative solution: We again consider a megastep. The first (topmost) card goes to the 51st place; the 51st card goes to the 76th place, etc. This yields a so-called cycle, which we denote as $1 \to 51 \to 76 \to 13 \to \cdots \to 1$. From this notation, we deduce for example that after two megasteps, the first card moves to the 76th place. If the first card returns to its original place after k megasteps, we say that the length of the cycle is k. It's now clear that the 51st card also returns to the 51st place after k megasteps. In this way, we can describe the behavior of the cards in a megastep, with a number of cycles. Now if we take a common multiple of the lengths of the cycles, all cards will return to their original places. $\square$

BxMO Team Selection Test, March 2010

Problems

1 Let $ABCD$ be a trapezoid with $AB \parallel CD$, $2|AB| = |CD|$ and $BD \perp BC$. Let M be the midpoint of CD and let E be the intersection BC and AD. Let C be the intersection of AM and BD. Let N be the intersection of OE and AB.

(a) Prove that $ABMD$ is a rhombus.

(b) Prove that the line DN passes through the midpoint of the line segment BE.

2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)f(y) = f(x+y) + xy$$

for all $x, y \in \mathbb{R}$.

3 Let N be the number of ordered 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ of positive integers satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} = 1.$$

Is N even or odd?

4 The two circles Γ_1 and Γ_2 intersect at P and Q. The common tangent that's on the same side as P, intersects the circles at A and B, respectively. Let C be the second intersection with Γ_2 of the tangent to Γ_1 at P, and let D be the second intersection with Γ_1 of the tangent to Γ_2 at Q. Let E be the intersection of AP and BC, and let F be the intersection of BP and AD. Let M be the image of P under point reflection with respect to the midpoint of AB. Prove that $AMBEQF$ is a cyclic hexagon.

5 For any non-negative integer n, we say that a permutation $(a_0, a_1, \ldots, a_n)$ of $\{0, 1, \ldots, n\}$ is *quadratic* if $k + a_k$ is a square for $k = 0, 1, \ldots, n$. Show that for any non-negative integer n, there exists a quadratic permutation of $\{0, 1, \ldots, n\}$.

Solutions

1 From $2|AB| = |CD|$ and $AB \parallel CD$, we deduce that AB is a midsegment in triangle CDE. Hence A is the midpoint of DE. Since $\angle DBE = 90°$, by Thales' Theorem, A is the center of the circle passing through D, B and E. Hence $|AD| = |AE| = |AB|$. We already know that $|AB| = |DM| = |MC|$. In a similar way, using Thales' Theorem, we obtain $|BM| = |CM| = |DM|$. Hence all sides of the quadrilateral $ABMD$ have the same length; $ABMD$ is a rhombus (a). The intersection of the diagonals of a rhombus is the midpoint of both diagonals, hence O is the midpoint of BD. Since A is the midpoint of DE, it follows that N is the centroid of triangle BDE. Hence DN passes through the midpoint of BE (b). $\square$

2 First, note that the function given by $f(x) = 0$ for all $x \in \mathbb{R}$, does not satisfy the functional equation. Hence there exists an $x_0 \in \mathbb{R}$ for which $f(x_0) \neq 0$. Substituting $x = x_0$ and $y = 0$ gives $f(x_0)f(0) = f(x_0)$. Since $f(x_0) \neq 0$, we obtain $f(0) = 1$. Now substituting $x = 1$ and $y = -1$ gives $f(1)f(-1) = f(0) - 1 = 1 - 1 = 0$. Hence $f(1) = 0$ or $f(-1) = 0$.

We now distinguish the two cases. First, suppose that $f(1) = 0$, and substitute $x = 1$. We then get $0 = f(1)f(y) = f(1 + y) + y$ for all $y \in \mathbb{R}$, so $f(1 + y) = -y$ for all $y \in \mathbb{R}$. Substituting $y = t - 1$ now gives $f(t) = -t + 1$ for all $t \in \mathbb{R}$. This is our first candidate solution.

Now suppose that $f(-1) = 0$ and substitute $x = -1$. We obtain $0 = f(-1)f(y) = f(-1 + y) - y$ for all $y \in \mathbb{R}$, hence $f(-1 + y) = y$ for all $y \in \mathbb{R}$. Substituting $y = t + 1$, we obtain $f(t) = t + 1$ for all $t \in \mathbb{R}$. This is our second candidate solution.

Since we have covered all the cases, these are the only two possible solutions. Let's check them.

If $f(t) = -t + 1$ for all $t \in \mathbb{R}$, then for all $x, y \in \mathbb{R}$:

$$f(x)f(y) = (-x + 1)(-y + 1) = (-x - y + 1) + xy = f(x + y) + xy.$$

If $f(t) = t + 1$ for all $t \in \mathbb{R}$, then for all $x, y \in \mathbb{R}$:

$$f(x)f(y) = (x + 1)(y + 1) = (x + y + 1) + xy = f(x + y) + xy.$$

Hence both candidate solutions are indeed solutions of the functional equation. $\square$

3 **Solution I.** Consider an unordered 5-tuple that satisfies the given equation, and suppose that it consists of the distinct integers $b_1, \ldots, b_k$, where b_i occurs exactly t_i times. Hence $t_1 + \cdots + t_k = 5$. Note that this unordered 5-tuple can be ordered in $\frac{5!}{t_1! \cdots t_k!}$ ways. This is odd if and only if three factors 2 occur in the denominator, which holds if and only if 4! or 5! occurs in the denominator. Hence the only unordered 5-tuples that can be ordered in a odd number of ways, are those that consist of at least 4 numbers that are the same.

Such a 5-tuple has the form (a, a, a, a, b), where b can be equal to a, and where a, b are positive integers. We obtain the equation $\frac{4}{a} + \frac{1}{b} = 1$, or, equivalently, $4b + a = ab$. We deduce that b has to be a divisor of a and that a is a divisor of $4b$. Hence there are three possibilities: $a = b$, $a = 2b$ and $a = 4b$. In the first case, we obtain $5b = b^2$, hence $b = 5$, yielding the solution $(5, 5, 5, 5, 5)$. In the second case, we obtain $6b = 2b^2$, hence $b = 3$, yielding the solution $(6, 6, 6, 6, 3)$. In the third and final case, we obtain $8b = 4b^2$, so $b = 2$, yielding the solution $(8, 8, 8, 8, 2)$. Hence there are three unordered 5-tuples which can be ordered in an odd number of ways. Hence N is odd. $\square$

Solution II. Let $(a_1, a_2, a_3, a_4, a_5)$ be an ordered 5-tuple that satisfies the given equation. Then also the 5-tuple $(a_2, a_1, a_3, a_4, a_5)$. If $a_1 \neq a_2$, this is a different ordered 5-tuple. Hence there is an even number of solutions $(a_1, a_2, a_3, a_4, a_5)$ with $a_1 \neq a_2$. So we may only consider the solutions with $a_1 = a_2$. In the same way, we show that there is an even number of solutions $(a_1, a_1, a_3, a_4, a_5)$ with $a_3 \neq a_4$. Hence we may only consider the solutions with $a_1 = a_2$, $a_3 = a_4$. For each solution $(a_1, a_1, a_3, a_3, a_5)$ with $a_1 \neq a_3$, there is yet another solution $(a_3, a_3, a_1, a_1, a_5)$, hence there is an even number of solutions $(a_1, a_1, a_3, a_3, a_5)$ with $a_1 \neq a_3$. Thus we may only consider the solution of the form (a, a, a, a, b), where b can be equal to a.

We obtain the equation $\frac{4}{a} + \frac{1}{b} = 1$, or equivalently, $4b + a = ab$. We can rewrite this last equation as $(a - 4)(b - 1) = 4$. Since b is a positive integer, we have $b - 1 \geq 0$. Hence $b - 1$ is a positive divisor of 4, namely 1, 2 or 4. This yields the three solutions $(a, b) = (8, 2)$, $(a, b) = (6, 3)$ and $(a, b) = (5, 5)$. Since the number of solutions that we covered previously, is even, we deduce that N is odd. $\square$

4 We express all relevant angles in terms of $\alpha = \angle BAP$ and $\beta = \angle PBA$. The midpoint of AB is, by definition, also the midpoint of PM, hence the diagonals of quadrilateral divide each other into equal segments. Hence $APBM$ is a parallelogram and we have $\angle AMB = \angle APB$. Note that (using the sum of the angles of a triangle) $\angle APB + \alpha + \beta = 180°$. Hence $180° - \angle AMB = \alpha + \beta$.

By the inscribed angle theorem, applied on the tangent AB to Γ_1, we have $\alpha = \angle ADP$. Then, by the inscribed angle theorem, applied on the chord AP of Γ_1, it follows that $\angle ADP = \angle AQP$. Hence $\alpha = \angle AQP$. In a similar way, we deduce that $\beta = \angle PCB = \angle PQB$. Hence $\angle AQB = \angle AQP + \angle PQB = \alpha + \beta = 180° - \angle AMB$, so $AQBM$ is a cyclic quadrilateral.

Now let S be the intersection of DP and AB. Using the inscribed angle theorem, we deduce that $\angle SPB = \angle PCB = \angle PBS = \angle PBA = \beta$. Hence $\angle DPF = \angle SPB = \beta$, by opposite angles. Now we apply the exterior angle theorem to $\triangle DFP$, to obtain $\angle AFB = \angle AFP = \angle FDP + \angle DPF = \alpha + \beta = \angle AQB$. Hence $AFQB$ is a cyclic quadrilateral. Hence F lies on the circumcircle of the cyclic quadrilateral $AQBM$. Similarly, we see that E also lies on that circle. Hence $AMBEQF$ is a cyclic hexagon. $\square$

5 We use induction on n. For $n = 0$ the permutation (0) is quadratic, since $0 + 0$ is a square.

Now let $l \geq 0$ and assume that for every $n \leq l$, there exists a quadratic permutation (induction hypothesis). Consider $n = l + 1$. Let m be such that m^2 is the least square greater than or equal to $l + 1$. Now we have $l \geq (m-1)^2 = m^2 - 2m + 1$, so $2(l+1) \geq 2m^2 - 4m + 4 = m^2 + (m-2)^2 \geq m^2$. We deduce that there exists an integer p with $0 \leq p \leq l + 1$ such that $(l + 1) + p = m^2$.

Now define a permutation as follows. If $p \geq 1$, then we take $(a_0, a_1, \ldots, a_{p-1})$ a quadratic permutation of $\{0, 1, \ldots, p-1\}$; which exists, by our induction hypothesis, since $p - 1 \leq l$. For $p \leq i \leq l+1$, we define $a_i = m^2 - i$. (Note that for $p = 0$, we now also have define a permutation of $(a_0, a_1, \ldots, a_{l+1})$.)

Now the sequence $(a_p, a_{p+1}, \ldots, a_{l+1})$ is exactly $(l+1, l, \ldots, p)$, which implies that, together with the initial part, we have used all values from 0 up to $l + 1$. Also, $a_i + i$ is a square for all i. Hence $(a_0, a_1, \ldots, a_{l+1})$ is a quadratic permutation of $\{0, 1, \ldots, l+1\}$. $\square$

Benelux Mathematical Olympiad, April 2010

Problems

1. A finite set of integers is called *bad* if its elements add up to 2010. A finite set of integers is a *Benelux-set* if none of its subsets is bad. Determine the smallest integer n such that the set $\{502, 503, 504, \ldots, 2009\}$ can be partitioned into n Benelux-sets.
(A partition of a set S into n subsets is a collection of n pairwise disjoint subsets of S, the union of which equals S.)

2. Find all polynomials $p(x)$ with real coefficients such that

$$p(a+b-2c)+p(b+c-2a)+p(c+a-2b) = 3p(a-b)+3p(b-c)+3p(c-a)$$

for all $a, b, c \in \mathbb{R}$.

3. On a line l there are three different points A, B and P in that order. Let a be the line through A perpendicular to l, and let b be the line through B perpendicular to l. A line through P, not coinciding with l, intersects a in Q and b in R. The line through A perpendicular to BQ intersects BQ in L and BR in T. The line through B perpendicular to AR intersects AR in K and AQ in S.

(a) Prove that P, T, S are collinear.

(b) Prove that P, K, L are collinear.

4. Find all quadruples (a, b, p, n) of positive integers, such that p is a prime and

$$a^3 + b^3 = p^n.$$

Solutions

1. As $502 + 1508 = 2010$, the set $S = \{502, 503, \ldots, 2009\}$ is not a Benelux-set, so $n = 1$ does not work. We will prove that $n = 2$ does work, i.e. that S can be partitioned into 2 Benelux-sets.

Define the following subsets of S:

$$\begin{aligned} A &= \{502, 503, \ldots, 670\}, \\ B &= \{671, 672, \ldots, 1005\}, \\ C &= \{1006, 1007, \ldots, 1339\}, \\ D &= \{1340, 1341, \ldots, 1508\}, \\ E &= \{1509, 1510, \ldots, 2009\}. \end{aligned}$$

We will show that $A \cup C \cup E$ and $B \cup D$ are both Benelux-sets.

Note that there does not exist a bad subset of S of one element, since that element would have to be 2010. Also, there does not exist a bad subset of S of more than three elements, since the sum of four or more elements would be at least $502 + 503 + 504 + 505 = 2014 > 2010$. So any possible bad subset of S contains two or three elements.

Consider a bad subset of two elements a and b. As $a, b \geq 502$ and $a + b = 2010$, we have $a, b \leq 2010 - 502 = 1508$. Furthermore, exactly one of a and b is smaller than 1005 and one is larger than 1005. So one of them, say a, is an element of $A \cup B$, and the other is an element of $C \cup D$. Suppose $a \in A$, then $b \geq 2010 - 670 = 1340$, so $b \in D$. On the other hand, suppose $a \in B$, then $b \leq 2010 - 671 = 1339$, so $b \in C$. Hence $\{a, b\}$ cannot be a subset of $A \cup C \cup E$, nor of $B \cup D$.

Now consider a bad subset of three elements a, b and c. As $a, b, c \geq 502$, $a + b + c = 2010$, and the three elements are pairwise distinct, we have $a, b, c \leq 2010 - 502 - 503 = 1005$. So $a, b, c \in A \cup B$. At least one of the elements, say a, is smaller than $\frac{2010}{3} = 670$, and at least one of the elements, say b, is larger than 670. So $a \in A$ and $b \in B$. We conclude that $\{a, b, c\}$ cannot be a subset of $A \cup C \cup E$, nor of $B \cup D$.

This proves that $A \cup C \cup E$ and $B \cup D$ are Benelux-sets, and therefore the smallest n for which S can be partitioned into n Benelux-sets is $n = 2$. $\square$

Remark. Observe that $A \cup C \cup E_1$ and $B \cup D \cup E_2$ are also Benelux-sets, where $\{E_1, E_2\}$ is any partition of E.)

2. For $a = b = c$, we have $3p(0) = 9p(0)$, hence $p(0) = 0$. Now set $b = c = 0$, then we have

$$p(a) + p(-2a) + p(a) = 3p(a) + 3p(-a)$$

for all $a \in \mathbb{R}$. So we find a polynomial equation

$$p(-2x) = p(x) + 3p(-x). \tag{1}$$

Note that the zero polynomial is a solution to this equation. Now suppose that p is not the zero polynomial, and let $n \geq 0$ be the degree of p. Let $a_n \neq 0$ be the coefficient of x^n in $p(x)$. At the left-hand side of (1), the coefficient of x^n is $(-2)^n \cdot a_n$, while at the right-hand side the coefficient of x^n is $a_n + 3 \cdot (-1)^n \cdot a_n$. Hence $(-2)^n = 1 + 3 \cdot (-1)^n$. For n even, we find $2^n = 4$, so $n = 2$, and for n odd, we find $-2^n = -2$, so $n = 1$. As we already know that $p(0) = 0$, we must have $p(x) = a_2x^2 + a_1x$, where a_1 and a_2 are real numbers (possibly zero).

The polynomial $p(x) = x$ is a solution to our problem, as

$$(a+b-2c)+(b+c-2a)+(c+a-2b) = 0 = 3(a-b)+3(b-c)+3(c-a)$$

for all $a, b, c \in \mathbb{R}$. Also, $p(x) = x^2$ is a solution, since

$$\begin{aligned}(a+b-2c)^2 + (b+c-2a)^2 + (c+a-2b)^2 \\ &= 6(a^2+b^2+c^2) - 6(ab+bc+ca) \\ &= 3(a-b)^2 + 3(b-c)^2 + 3(c-a)^2\end{aligned}$$

for all $a, b, c \in \mathbb{R}$.

Now note that if $p(x)$ is a solution to our problem, then so is $\lambda p(x)$ for all $\lambda \in \mathbb{R}$. Also, if $p(x)$ and $q(x)$ are both solutions, then so is $p(x) + q(x)$. We conclude that for all real numbers a_2 and a_1 the polynomial $a_2x^2 + a_1x$ is a solution. Since we have already shown that there can be no other solutions, these are the only solutions. $\square$

3. Solution 1.

(a) Since P, R and Q are collinear, we have $\triangle PAQ \sim \triangle PBR$, hence

$$\frac{|AQ|}{|BR|} = \frac{|AP|}{|BP|}.$$

Conversely, P, T and S are collinear if it holds that

$$\frac{|AS|}{|BT|} = \frac{|AP|}{|BP|}.$$

So it suffices to prove

$$\frac{|BT|}{|BR|} = \frac{|AS|}{|AQ|}.$$

Since $\angle ABT = 90^\circ = \angle ALB$ and $\angle TAB = \angle BAL$, we have $\triangle ABT \sim \triangle ALB$. And since $\angle ALB = 90^\circ = \angle QAB$ and $\angle LBA = \angle ABQ$, we have $\triangle ALB \sim \triangle QAB$. Hence $\triangle ABT \sim \triangle QAB$, so

$$\frac{|BT|}{|BA|} = \frac{|AB|}{|AQ|}.$$

Similarly, we have $\triangle ABR \sim \triangle AKB \sim \triangle SAB$, so

$$\frac{|BR|}{|BA|} = \frac{|AB|}{|AS|}.$$

Combining both results, we get

$$\frac{|BT|}{|BR|} = \frac{|BT|/|BA|}{|BR|/|BA|} = \frac{|AB|/|AQ|}{|AB|/|AS|} = \frac{|AS|}{|AQ|},$$

which had to be proved.

(b) Let the line PK intersect BR in B_1 and AQ in A_1 and let the line PL intersect BR in B_2 and AQ in A_2. Consider the points A_1, A and S on the line AQ, and the points B_1, B and T on the line BR. As $AQ \parallel BR$ and the three lines A_1B_1, AB and ST are concurrent (in P), we have

$$A_1A : AS = B_1B : BT,$$

where all lengths are directed. Similarly, as A_1B_1, AR and SB are concurrent (in K), we have

$$A_1A : AS = B_1R : RB.$$

This gives

$$\frac{BB_1}{BT} = \frac{RB_1}{RB} = \frac{RB + BB_1}{RB} = 1 + \frac{BB_1}{RB} = 1 - \frac{BB_1}{BR},$$

so

$$BB_1 = \frac{1}{\frac{1}{BT} + \frac{1}{BR}}.$$

Similary, using the lines A_2B_2, AB and QR (concurrent in P) and the lines A_2B_2, AT and QB (concurrent in L), we find

$$B_2B : BR = A_2A : AQ = B_2T : TB.$$

This gives

$$\frac{BB_2}{BR} = \frac{TB_2}{TB} = \frac{TB + BB_2}{TB} = 1 + \frac{BB_2}{TB} = 1 - \frac{BB_2}{BT},$$

so

$$BB_2 = \frac{1}{\frac{1}{BR} + \frac{1}{BT}}.$$

We conclude that $B_1 = B_2$, which implies that P, K and L are collinear.

□

Solution 2. As $\angle AKB = \angle ALB = 90°$, the points K and L belong to the circle with diameter AB. Since $\angle QAB = \angle ABR = 90°$, the lines AQ and BR are tangents to this circle.

Apply Pascal's theorem to the points A, A, K, L, B and B, all on the same circle. This yields that the intersection Q of the tangent in A and the line BL, the intersection R of the tangent in B and the line AK, and the intersection of KL and AB are collinear. So KL passes through the intersection of AB and QR, which is point P. Hence P, K and L are collinear. This proves part b.

Now apply Pascal's theorem to the points A, A, L, K, B and B. This yields that the intersection S of the tangent in A and the line BK, the intersection T of the tangent in B and the line AL, and the intersection P of KL and AB are collinear. This proves part a. □

4. Let (a, b, p, n) be a solution. Note that we can write the given equation as

$$(a+b)(a^2 - ab + b^2) = p^n.$$

As a and b are positive integers, we have $a + b \geq 2$, so $p \mid a + b$. Furthermore, $a^2 - ab + b^2 = (a-b)^2 + ab$, so either $a = b = 1$ or $a^2 - ab + b^2 \geq 2$. Assume that the latter is the case. Then p is a divisor of both $a+b$ and $a^2 - ab + b^2$, hence also of $(a+b)^2 - (a^2 - ab + b^2) = 3ab$. This means that p either is equal to 3 or is a divisor of ab. Since p is a divisor of $a + b$, we have $p \mid a \Leftrightarrow p \mid b$, hence either $p = 3$, or $p \mid a$ and $p \mid b$. If $p \mid a$ and $p \mid b$, then we can write $a = pa'$, $b = pb'$ with a' and b' positive integers, and we have $(a')^3 + (b')^3 = p^{n-3}$, so $(a', b', p, n-3)$ then is another solution (note that $(a')^3 + (b')^3$ is a positive integer greater than 1, so $n - 3$ is positive).

Now assume that (a_0, b_0, p_0, n_0) is a solution such that $p \nmid a$. From the reasoning above it follows that either $a_0 = b_0 = 1$, or $p_0 = 3$. After all, if we do not have $a_0 = b_0 = 1$ and we have $p_0 \neq 3$, then $p \mid a$. Also, given an arbitrary solution (a, b, p, n), we can divide everything by p repeatedly until there are no factors p left in a.

Suppose $a_0 = b_0 = 1$. Then the solution is $(1, 1, 2, 1)$.

Suppose $p_0 = 3$. Assume that $3^2 \mid (a_0^2 - a_0 b_0 + b_0^2)$. As $3^2 \mid (a_0 + b_0)^2$, we then have $3^2 \mid (a_0 + b_0)^2 - (a_0^2 - a_0 b_0 + b_0^2) = 3a_0 b_0$, so $3 \mid a_0 b_0$. But $3 \nmid a_0$ by assumption, and $3 \mid a_0 + b_0$, so $3 \nmid b_0$, which contradicts $3 \mid a_0 b_0$. We conclude that $3^2 \nmid (a_0^2 - a_0 b_0 + b_0^2)$. As both $a_0 + b_0$ and $a_0^2 - a_0 b_0 + b_0^2$ must be powers of 3, we have $a_0^2 - a_0 b_0 + b_0^2 = 3$. Hence $(a_0 - b_0)^2 + a_0 b_0 = 3$. We must have $(a_0 - b_0)^2 = 0$ or $(a_0 - b_0)^2 = 1$. The former does not give a solution; the latter gives $a_0 = 2$ and $b_0 = 1$ or $a_0 = 1$ and $b_0 = 2$.

So all solutions with $p \nmid a$ are $(1, 1, 2, 1)$, $(2, 1, 3, 2)$ and $(1, 2, 3, 2)$. From the above it follows that all other solutions are of the form $(p_0^k a_0,\ p_0^k b_0,\ p_0,\ n_0 + 3k)$, where (a_0, b_0, p_0, n_0) is one of these three solutions. Hence we find three families of solutions:

- $(2^k,\ 2^k,\ 2,\ 3k+1)$ with $k \in \mathbb{Z}_{\geq 0}$,
- $(2 \cdot 3^k,\ 3^k,\ 3,\ 3k+2)$ with $k \in \mathbb{Z}_{\geq 0}$,
- $(3^k,\ 2 \cdot 3^k,\ 3,\ 3k+2)$ with $k \in \mathbb{Z}_{\geq 0}$.

It is easy to check that all these quadruples are indeed solutions. $\square$

IMO Team Selection Test 1, June 2010

Problems

1 Let ABC be an acute triangle such that $\angle BAC = 45°$. Let D a point on AB such that $CD \perp AB$. Let P be an internal point of the segment CD. Prove that $AP \perp BC$ if and only if $|AP| = |BC|$.

2 Let A and B be positive integers. Define the arithmetic sequence $a_0, a_1, a_2, \ldots$ by $a_n = An + B$. Suppose that there exists an $n \geq 0$ such that a_n is a square. Let M be a positive integer such that M^2 is the smallest square in the sequence. Prove that $M < A + \sqrt{B}$.

3 Let $n \geq 2$ be a positive integer and p a prime such that $n \mid p - 1$ and $p \mid n^3 - 1$. Show that $4p - 3$ is a square.

4 Let $ABCD$ be a cyclic quadrilateral satisfying $\angle ABD = \angle DBC$. Let E be the intersection of the diagonals AC and BD. Let M be the midpoint of AE, and N be the midpoint of DC. Show that $MBCN$ is a cyclic quadrilateral.

5 Find all triples (x, y, z) of real (but not necessarily positive) numbers satisfying

$$3(x^2 + y^2 + z^2) = 1,$$
$$x^2y^2 + y^2z^2 + z^2x^2 = xyz(x + y + z)^3.$$

Solutions

1 Let E be the intersection of AP and BC. Note that $\angle DCA = 90° - \angle CAD = 90° - \angle CAB = 45°$, so $\triangle ACD$ is isosceles: $|AD| = |CD|$.

Now suppose that $|AP| = |BC|$. Since $\angle ADP = 90° = \angle CDB$, we have $\triangle ADP \cong \triangle CDB$, by (SSR). Hence $\angle APD = \angle CBD$, from which follows that

$$\begin{aligned}\angle CEA = \angle CEP &= 180° - \angle EPC - \angle PCE \\ &= 180° - \angle APD - \angle DCB = 180° - \angle CBD - \angle DCB \\ &= \angle BDC = 90°,\end{aligned}$$

hence $AP \perp BC$.

Conversely, suppose that $AP \perp BC$, or equivalently, $\angle CEP = 90°$. Then we have

$$\angle APD = \angle EPC = 90° - \angle PCE = 90° - \angle DCB = \angle CBD.$$

Since we also have $\angle ADP = 90° = \angle CDB$, from (SAA), it follows that $\triangle ADP \cong \triangle CDB$. Hence $|AP| = |BC|$. $\square$

2 If $M \leq A$, then automatically, $M < A + \sqrt{B}$, so we're done in this case.

Hence suppose that $M > A$. Let k be such that $a_k = M^2$. So $Ak + B = M^2$. Since $0 < M - A < M$, the square $(M-A)^2$ is smaller than M^2. We have $(M-A)^2 = M^2 - 2MA + A^2 = M^2 - A(2M - A)$. So if $k - (2M - A) \geq 0$, then

$$\begin{aligned} a_{k-(2M-A)} &= A(k - (2M - A)) + B = (Ak + B) - 2MA + A^2 \\ &= M^2 - 2MA + A^2 = (M - A)^2, \end{aligned}$$

and in that case, M^2 is not the smallest square in the sequence, contrary to our assumption. Hence $k - (2M - A) < 0$. From this, it follows that $A(k-(2M-A))+B < B$, or equivalently, $(M-A)^2 < B$. Since $M - A$ is positive, we deduce that $M - A < \sqrt{B}$, hence that $M < A + \sqrt{B}$. $\square$

3 **Solution I.** From $n \mid p - 1$, we deduce that $n < p$. Since p is prime, p divides one of the two factors of $n^3 - 1 = (n-1)(n^2+n+1)$, but p is too large to be a divisor of $n - 1 > 0$. Hence $p \mid n^2 + n + 1$. Since $n \mid p-1$, we can write p as $kn+1$, where k is a positive integer. Since $kn + 1 \mid n^2 + n + 1$, we have

$$kn + 1 \mid k(n^2 + n + 1) - (n + 1)(kn + 1) = k - n - 1.$$

We now distinguish three cases.

Case 1: $k > n + 1$. In this case, $k - n - 1$ is positive, and we must have $kn + 1 \leq k - n - 1$. It follows that $(k + 1)n \leq k - 2$. But the left hand side is clearly larger than the right hand side, yielding a contradiction.

Case 2: $k < n + 1$. In this case, $k - n - 1$ is negative, and we must have $kn + 1 \leq n + 1 - k$. It follows that $(k - 1)n \leq -k$. But the

left hand side is non-negative, and the right hand side is negative, yielding a contradiction.

Case 3: $k = n + 1$. Now we have $p = (n + 1)n + 1 = n^2 + n + 1$. Hence $4p - 3 = 4n^2 + 4n + 1 = (2n + 1)^2$ is a square.

We deduce that in the only possible case, $4p - 3$ is a square. $\square$

Solution II. As in the first solution, we show that $p \mid n^2 + n + 1$. If $p = n^2 + n + 1$, then we have $4p - 3 = 4n^2 + 4n + 1 = (2n + 1)^2$, which is a square. Now suppose that $p \neq n^2 + n + 1$. Then there is an integer $m > 1$ such that $pm = n^2 + n + 1$. Reducing modulo n, we get $pm \equiv 1 \mod n$. From $n \mid p - 1$, we deduce that $p \equiv 1 \mod n$, hence $m \equiv 1 \mod n$. So p and m are at least $n + 1$. However, $(n + 1)^2 = n^2 + 2n + 1 > n^2 + n + 1 = pm$, yielding a contradiction. $\square$

4 **Solution I.** Since $ABCD$ is a cyclic quadrilateral, we have $\angle BDC = \angle BAC = \angle BAE$. Also, from the given properties, it follows that $\angle CBD = \angle EBA$. Hence we have, by (aa), $\triangle DCB \sim \triangle AEB$. We now show that this implies that $\triangle NCB \sim \triangle MEB$.

First of all, we have $\angle NCB = \angle DCB = \angle AEB = \angle MEB$. Also, we have $\frac{|NC|}{|ME|} = \frac{2|NC|}{2|ME|} = \frac{|DC|}{|AE|} = \frac{|CB|}{|EB|}$. By (sas), we deduce that indeed $\triangle NCB \sim \triangle MEB$. Hence $\angle BMC = \angle BME = \angle BNC$, and it follows that $MBCN$ is a cyclic quadrilateral. $\square$

Solution II. As in the first solution, we show that $\triangle DCB \sim \triangle AEB$. Now note that the lines BM and BN are median in these two similar triangles. Hence the corresponding angles $\angle BME$ and $\angle BNC$ are equal. Hence $\angle BMC = \angle BME = \angle BNC$, from which we deduce that $MBCN$ is a cyclic quadrilateral. $\square$

5 We first show that for reals x, y and z satisfying the first condition, we have

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z)^3. \tag{2}$$

So the solutions that we are looking for, are the cases in which equality holds in 2.

Let a, b and c be reals. Then $(a-b)^2 \geq 0$, so $\frac{a^2+b^2}{2} \geq ab$ with equality if and only if $a = b$. We repeat this for the pairs (b, c) and (c, a), and after summing, it follows that

$$a^2 + b^2 + c^2 \;\geq\; ab + bc + ca$$

with equality if and only if $a = b = c$.

We apply this inequality to the triple (xy, yz, zx), and obtain

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xy^2z + yz^2x + zx^2y = xyz(x + y + z)$$

with equality if and only if $xy = yz = zx$. We also apply it to (x, y, z) to obtain $x^2 + y^2 + z^2 \geq xy + yz + zx$. Hence from the first condition, it follows that

$$\begin{aligned} 1 = 3(x^2 + y^2 + z^2) &= (x^2 + y^2 + z^2) + 2(x^2 + y^2 + z^2) \\ &\geq x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2, \end{aligned}$$

with equality if and only if $x = y = z$. Now we're going to combine these two inequalities,

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z) \tag{3}$$

and

$$1 \geq (x + y + z)^2 \tag{4}$$

To this end, we first multiply (4) by the non-negative factor $x^2y^2 + y^2z^2 + z^2x^2$; we obtain

$$(x^2y^2 + y^2z^2 + z^2x^2) \cdot 1 \geq (x^2y^2 + y^2z^2 + z^2x^2) \cdot (x + y + z)^2$$

with equality if and only if $x = y = z$ or $x^2y^2 + y^2z^2 + z^2x^2 = 0$. Next, we multiply (3) by the non-negative factor $(x + y + z)^2$, from which follows that

$$(x^2y^2 + y^2z^2 + z^2x^2) \cdot (x + y + z)^2 \geq xyz(x + y + z) \cdot (x + y + z)^2$$

with equality if and only if $xy = yz = zx$ or $x + y + z = 0$. Combining these yields

$$\begin{aligned} (x^2y^2 + y^2z^2 + z^2x^2) \cdot 1 \ &\geq \ (x^2y^2 + y^2z^2 + z^2x^2) \cdot (x + y + z)^2 \\ &\geq \ xyz(x + y + z) \cdot (x + y + z)^2, \end{aligned}$$

with equality if and only if

$$\begin{aligned} &\big(x = y = z \text{ or } x^2y^2 + y^2z^2 + z^2x^2 = 0\big) \\ \text{and } &\big(xy = yz = zx \text{ or } x + y + z = 0\big)\,. \end{aligned}$$

From the second condition, it follows that indeed, equality must hold. Hence (x, y, z) must satisfy the conditions above. Furthermore, we

still have the first condition, which says that $3(x^2+y^2+z^2)=1$:

$$\begin{aligned} 3(x^2+y^2+z^2)&=1,\\ x=y=z \quad &\text{or} \quad x^2y^2+y^2z^2+z^2x^2=0,\\ xy=yz=zx \quad &\text{or} \quad x+y+z=0. \end{aligned}$$

We distinguish two cases. First, suppose that $x=y=z$. Then we also have $xy=yz=xz$. Using the first condition, we obtain $1=3(x^2+y^2+z^2)=9x^2=(3x)^2$, hence $x=y=z=\pm\frac{1}{3}$.

Next, suppose that $x^2y^2+y^2z^2+z^2x^2=0$; then each term must be zero, hence $xy=yz=zx=0$. From $xy=0$, we deduce without loss of generality that $x=0$, and also, from $yz=0$, we deduce without loss of generality that $y=0$. In this way, we find the solutions $(0,0,\pm\frac{1}{3}\sqrt{3})$, and also the solutions $(0,\pm\frac{1}{3}\sqrt{3},0)$ and $(\pm\frac{1}{3}\sqrt{3},0,0)$.$\square$

Remark. Once you've shown (3) and (4), you can also obtain (2) by multiplying (4) by $xyz(x+y+z)$, but you'll need to show that $xyz(x+y+z)$ is non-negative. From the second condition, we deduce that $xyz(x+y+z)^3$ is a sum of squares, hence non-negative. If $x+y+z\neq 0$, it follows that $xyz(x+y+z)\geq 0$, and otherwise, we have $xyz(x+y+z)=0$. Hence indeed, in both cases we have $xyz(x+y+z)\geq 0$.

Using this method, the equality case becomes:

$$\begin{aligned} 3(x^2+y^2+z^2)&=1,\\ xy=yz&=zx,\\ x=y=z \quad &\text{or} \quad xyz(x+y+z)=0. \end{aligned}$$

IMO Team Selection Test 2, June 2010

Problems

1. Consider sequences a_1, a_2, a_3, ... of positive integers. Determine the smallest possible value of a_{2010} if

(i) $a_n < a_{n+1}$ for all $n \geq 1$,

(ii) $a_i + a_l > a_j + a_k$ for all quadruples (i, j, k, l) which satisfy $1 \leq i < j \leq k < l$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x) = \max_{y \in \mathbb{R}} (2xy - f(y))$$

for all $x \in \mathbb{R}$.

(In general the expression $a = \max_{s \in S} g(s)$ means: $a \geq g(s)$ for all $s \in S$ and furthermore there is an $s \in S$ for which $a = g(s)$.)

3. (a) Let a and b be positive integers such that $M(a, b) = a - \frac{1}{b} + b\left(b + \frac{3}{a}\right)$ is an integer. Proof that $M(a, b)$ is a square.

(b) Find nonzero integers a and b such that $M(a, b)$ is a positive integer, but not a square.

4. Let $ABCD$ be a square with circumcircle Γ_1. Let P be a point on the arch AC that also contains B. A circle Γ_2 touches Γ_1 in P and also touches the diagonal AC in Q. Let R be a point on Γ_2 such that the line DR touches Γ_2. Proof that $|DR| = |DA|$.

5. The polynomial $A(x) = x^2 + ax + b$ with integer coefficients has the following property: for each prime p there is an integer k such that $A(k)$ and $A(k + 1)$ are both divisible by p. Proof that there is an integer m such that $A(m) = A(m + 1) = 0$.

Solutions

1. By induction we prove that $a_n - a_1 \geq 2^{n-1} - 1$ for all $n \geq 2$. For $n = 2$ this reduces to $a_2 - a_1 \geq 1$ and this follows from condition (i). Let $m \geq 2$ and suppose that $a_m - a_1 \geq 2^{m-1} - 1$ (IH). We apply condition (ii) with $i = 1$, $j = k = m$ and $l = m + 1$. This yields $a_1 + a_{m+1} > 2a_m$. So

$$a_{m+1} - a_1 > 2a_m - 2a_1 \overset{\text{IH}}{\geq} 2(2^{m-1} - 1) = 2^m - 2,$$

and since a_{m+1} is a positive integer, this yields $a_{m+1} - a_2 \geq 2^m - 1$. This completes the induction.

We now know that for $n \geq 1$:

$$a_n \geq 2^{n-1} - 1 + a_1 \geq 2^{n-1}$$

and in particular $a_{2010} \geq 2^{2009}$.

On the other hand we prove that a_{2010} is possible by showing that the sequence given by $a_n = 2^{n-1}$ satisfies the conditions. This sequence consists of positive integers and is strictly increasing (condition (i)). Let (i, j, k, l) be a quadruple that satisfies $1 \leq i < j \leq k < l$. It is true that

$$a_j + a_k = 2^{j-1} + 2^{k-1} \leq 2^{k-1} + 2^{k-1} = 2^k \leq 2^{l-1} < 2^{i-1} + 2^{l-1},$$

and thus the sequence also satisfies condition (ii).

We conclude that the smallest possible value of a_{2010} is equal to 2^{2009}. $\square$

2. For all $x \in \mathbb{R}$ it is true that

$$f(x) = \max_{y \in \mathbb{R}} (2xy - f(y)) \geq 2x^2 - f(x),$$

so $2f(x) \geq 2x^2$, that is $f(x) \geq x^2$.

Because $(x - y)^2 \geq 0$, it is true that $x^2 \geq 2xy - y^2$ for all $x, y \in \mathbb{R}$. Because we already have showed that $f(y) \geq y^2$, we know that $2xy - f(y) \leq 2xy - y^2 \leq x^2$ and consequently

$$f(x) = \max_{y \in \mathbb{R}} (2xy - f(y)) \leq x^2.$$

We conclude $f(x) = x^2$.

We check if this function satisfies. Let $x \in \mathbb{R}$ be given. Because $(x-y)^2 \geq 0$, it is true that $x^2 \geq 2xy - y^2$ for all $y \in \mathbb{R}$ with equality iff $x = y$, therefore

$$x^2 = \max_{y \in \mathbb{R}} (2xy - y^2).$$

□

3. (a) Because $a + b^2$ is an integer, $-\frac{1}{b} + \frac{3b}{a}$ is also an integer. We may write this as $\frac{-a+3b^2}{ab}$. We know that ab is a divisor of $3b^2 - a$. In particular b is a divisor of $3b^2 - a$ and therefore $b \mid a$. However, this means that b^2 is a divisor of ab and consequently it is also a divisor of $3b^2 - a$, which yields $b^2 \mid a$. We now write $a = mb^2$ with m a positive integer. Then is is true that mb^3 is a divisor of $3b^2 - mb^2$, so mb is a divisor of $3 - m$. This yields that m is a divisor of 3 (that is $m = 1$ or $m = 3$) and that b is a divisor of $3 - m$.

First suppose that $m = 3$. Then we have $a = 3b^2$. Filling this in yields $M(3b^2, b) = 3b^2 - \frac{1}{b} + b^2 + \frac{1}{b} = 4b^2$ and that is the square of $2b$.

Now suppose that $m = 1$. From $b \mid 3 - m$ follows $b = 1$ or $b = 2$. In the first case we have $a = 1$ and in the second case $a = 4$. Filling in the first possibility yields $M(1, 1) = 1 - 1 + 1 + 3 = 4$, which is a square. Filling in the second possibility yields $M(4, 2) = 4 - \frac{1}{2} + 4 + \frac{3}{2} = 0$, which also is a square.

We conclude that $M(a, b)$ is a square in all cases.

(b) Take $a = 4$ and $b = -2$. Then we have $M(4, -2) = 7$, which is a positive integer, but not a square.

After doing part (a) this answer is not hard to find. You know that a has to be equal to mb^2, but now for some $m \in \mathbb{Z}$, and that m has to be a divisor of 3. Further more $m = 3$ will not work, because it yields a square. You can try the other possibilities of m yourself.

□

4. Let M be the intersection of AC and BD (that is the midpoint of Γ_1) and let N be the midpoint of Γ_2. First we prove that P, Q and D are collinear.

If $P = B$, then $Q = M$ and it is trivial. If not, let S be the intersection of PQ and BD. We will first prove that $S = D$. Notice

that M, N and P are collinear and that QN is parallel to DB. This yields because of F-angles that $\angle PSB = \angle PQN = \angle NPQ$, because $|NP| = |NQ|$. By Z-angles we see that $\angle PMB = \angle MNQ$ and because of the exterior angle theorem inside triangle PQN that is equal to $\angle PQN + \angle NPQ = 2\angle PSB$. So $\angle PMB = \angle PSB$, together with the inscribed angle theorem this yields that S lies on Γ_1. So $S = D$, from which follows that P, Q and D are collinear.

Because $\angle DPB = \angle DMQ = 90°$ and $\angle PDB = \angle QDM$, it is true that $\triangle DPB \sim \triangle DMQ$. This yields $\frac{|DP|}{|DB|} = \frac{|DM|}{|DQ|}$. Since $|DB| = 2|DM|$, this is equivalent to $|DP||DQ| = 2|DM|^2$. The power theorem on Γ_2 yields $|DP||DQ| = |DR|^2$, so $|DR| = \sqrt{2}|DM| = |DA|$. $\square$

5. Let p be a prime and let k be such that $A(k)$ and $A(k+1)$ are both divisible by p. The difference between $A(k)$ and $A(k+1)$ is also divisible by p and this is equal to

$$A(k+1) - A(k) = (k+1)^2 + a(k+1) + b - (k^2 + ak + b) = 2k + 1 + a,$$

so $2k \equiv -1 - a \mod p$. Because $A(k)$ is divisible by p, the integer $4A(k)$ is also divisible by p, so we have modulo p that

$$\begin{aligned} 4A(k) &\equiv 4k^2 + 4ak + 4b \equiv (-1-a)^2 + 2(-1-a)a + 4b \\ &\equiv -a^2 + 4b + 1 \mod p. \end{aligned}$$

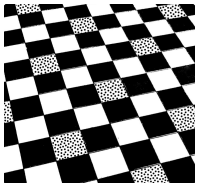
The right hand side is independent of k. We now see that for each prime p the number $-a^2 + 4b + 1$ has to be divisible by p. This is only possible if $-a^2 + 4b + 1 = 0$. So we have $a^2 = 4b + 1$. We now see that a has to be odd and we write $a = 2c + 1$ with $c \in \mathbb{Z}$. Then it is true that $4b + 1 = a^2 = 4c^2 + 4c + 1$, so $b = c^2 + c$. Therefore, the polynomial can be written as $A(x) = x^2 + (2c+1)x + (c^2 + c)$ for an integer c. We can factor this as $A(x) = (x+c)(x+c+1)$, consequently $x = -c$ and $x = -c - 1$ are zeroes of the polynomial. These are both integer points. We conclude that $m = -c - 1$ satisfies $A(m) = A(m+1) = 0$. $\square$

Junior Mathematical Olympiad, October 2009

Problems

Part 1

1. The station hall of Nijmegen is tiled in a repeating pattern with white, speckled and black tiles (see figure). Which part is speckled?

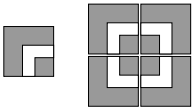


A) $\frac{1}{10}$ B) $\frac{1}{9}$ C) $\frac{1}{8}$ D) $\frac{1}{6}$ E) $\frac{1}{4}$

2. There are 120 chairs in a row. A number of chairs is already occupied in such a way that if someone comes in, this person can only sit on a chair next to someone else. What is the smallest amount of occupied chairs such that this may happen?

A) 40 B) 41 C) 59 D) 60 E) 80

3. Anne has got a lot of tiles like in the figure. She can lay down four of these tiles such that the white lines form a closed circuit. What is the smallest number of tiles with which she can lay down a larger circuit?

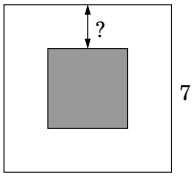


A) 6 B) 8 C) 9 D) 10 E) 12

4. Bram has got coins of 5 cent, 10 cent, 20 cent and 50 cent, of each type at least one. In total he has got 9 coins and together they are worth 2,10 euro. How many coins of 20 cent does Bram have?

A) 1 B) 2 C) 3 D) 4 E) 5

5. Exactly in the middle of a square with side length 7 a grey square is drawn. The area of the white part of the large square is three times as large as the area of the grey square. What is the width of the white ring?



A) 1 B) $1\frac{1}{3}$ C) $1\frac{1}{2}$ D) $1\frac{2}{3}$ E) $1\frac{3}{4}$

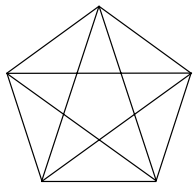
6. Alice, Bas, Chris, Daan and Eva know each other very well. Each of them always tells the truth or always lies. Chris says: "*Alice is honest*", on which Eve replies: "*Chris lies!*" Chris says: "*Bas is a liar.*" Eva claims: "*Daan is honest.*" Which two persons could both be telling the truth?

A) Alice and Bas B) Bas and Chris C) Chris and Daan
D) Daan and Eva E) Eva and Alice

7. Jan is looking for a triangle which has got circumference 21 and integer side lengths. Moreover, for each pair of sides the length of one of them has to divide the length of the other. How many different triangle with these properties exist?

A) 0 B) 1 C) 3 D) 5 E) 6

8. How many triangles are in this figure?



A) 20 B) 25 C) 30 D) 35 E) 40

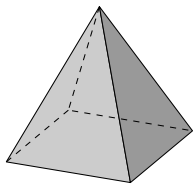
9. Vincent wants to tile a 1,30 by 1,70 meter terrace with 50 by 20 centimeter tiles. Because it is not possible to exactly tile the terrace he has to let some pieces of some tiles outside of the terrace. How many tiles does Vincent need to cover his terrace?

A) 22 B) 23 C) 24 D) 27 E) 28

10. A large bag contains a number of balls: more then 100, but less than 150. If you divide the balls (evenly) over 7 children, then 2 balls will remain. If you divide the balls over 6 children, 3 balls will remain. How many balls will remain if you divide the balls over 5 children?

A) 0 B) 1 C) 2 D) 3 E) 4

11. A pyramid with a square base is cut into two pieces by a straight cut. Which form can*not* be the cut?

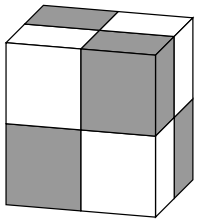


A) triangle B) square C) pentagon
D) hexagon E) All four forms are possible.

12. Mister Jansen departs every morning at 8:00 to his work place by car. If he (constantly) drives 40 km/h, he would arrive three minutes late at his work place. If he drives 60 km/h, he would arrive three minutes too early. How fast does he need to drive to be exactly on time?

A) 48 B) 49 C) 50 D) 51 E) 52

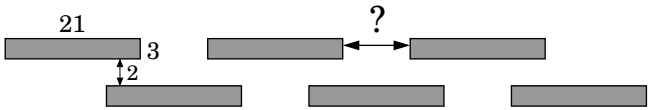
13. The $2 \times 2 \times 2$-cube in the figure has got four transparent cubes and four nontransparent cubes. You cannot look through the cube when looking from the front, side and upper face. A $3 \times 3 \times 3$-cube has got 27 cubes. What is the smallest amount of cubes you have to make nontransparent such that you can't look through the cube from the front, side and upper face?



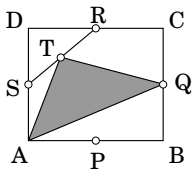
A) 9 B) 10 C) 12 D) 13 E) 14

14. Sir Tuinder has got a 'half-open' fence around his garden made of shelves. The shelves are alternating at the front and the rear side, exactly in front of the middle of the gap between the two shelves on the other side. Here you can see the fence from above. The width of the shelves is 21 cm and the thickness is 3 cm. Between the front- and the rear side is a distance of 2 cm. How many cm has the distance between two shelves next to each other be such that people you can't look into the garden from the outside?

A) 7 B) 8 C) 9 D) 10 E) 11



15. Rectangle $ABCD$ has got area 1. The points P, Q, R and S are the middles of the sides and T is the middle of RS. What is the area of triangle AQT?

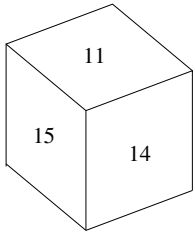


A) $\frac{1}{4}$ B) $\frac{3}{8}$ C) $\frac{5}{16}$ D) $\frac{9}{32}$ E) $\frac{17}{64}$

Part 2

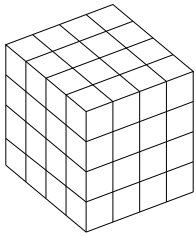
The answer to each problem is a number.

1. The numbers on the faces of this cube are six consecutive integers. If we add up the numbers on two opposing faces, we will always get the same answer. What is the sum of all six numbers?



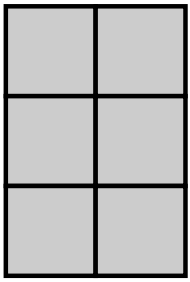
2. The entrance tickets of a amusement park normally cost 38 euro, but when it is a rainy day, you can go in for half of that price. Last week there 800 tickets were sold, together costing 19057 euro. How many tickets were sold for half the price?

3. In a $4 \times 4 \times 4$-cube each small cube has got 3, 4, 5 or 6 neighbors (that touch in a face to the small cube). What is the average number of neighbors that a small cube has got?

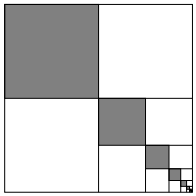


4. Jaap has got eight integers. Of each integer the first and last digits are 1 and the other digits are 0. The eight integers have got $2, 3, 5, 9, 17, 33, 65$ and 129 digits. If Jaap multiplies these eight digits and adds up the digit of the result, which number will he get?

5. A rectangular table has got three rows and two columns. On how many ways can you fill in the numbers 1 to 6 in the six positions in the table such that in each row the first number is greater than the second?

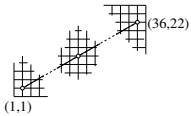


6. A square with area 54 is divided into four equal squares. The upper left square is colored grey; the lower right square is again divides into four equal squares, and so on. The pattern continues infinitely long. What is the area of the grey region?



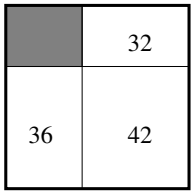
7. Start with a number of two digits (the first digit is nonzero) and do the following: multiply the two digits and add the same two digits to the product. In this way 27 will yield $14 + 2 + 7 = 23$. For how many two digit numbers the result is equal the the number, you began with?

8. A line segment is drawn in between the points $(1, 1)$ and $(36, 22)$. How many points with integer coordinates lie on this line segment, the end points included?



9. Anneke can drive from A to B on the high way or via a shorter road. On the high way Anneke can drive 120 km/h and on the shorter road she can drive 60 km/h. The high way is 8 km longer, but the trip takes 8 minutes less. How many kilometers long is the high way from A to B?

10. In the figure you see a square. The square is divided into four rectangles. Of three rectangles the *circumference* is given and written inside the rectangle. What is the *area* of the fourth rectangle?



Solutions

Part 1

1. C) $\frac{1}{8}$

2. A) 40

3. E) 12

4. A) 1

5. E) $1\frac{3}{4}$

6. D) Daan and Eva

7. C) 3

8. D) 35

9. B) 23

10. A) 0

11. D) hexagon

12. A) 48

13. A) 9

14. C) 9

15. C) $\frac{5}{16}$

Part 2

1. 81

2. 597

3. $4\frac{1}{2}$

4. $\underbrace{111\ldots111}_{256 \text{ ones}}$

5. 90

6. 18

7. 9

8. 8

9. 32

10. 42

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