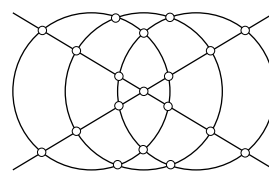# First Round
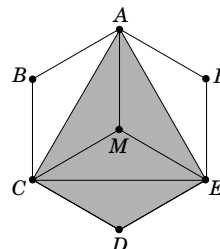# Dutch Mathematical Olympiad

Friday, Januari 29, 2010

Solutions

A1. D) 19 Any two circles intersect in at most 2 points, a circle and a line intersect in at most 2 points and two line intersect in at most 1 point. Therefore, the number of intersections cannot be more than $6 + 6 + 6 + 1 = 19$. You can easily check that this number of intersections can be realized, see for example the figure on the right.

A2. B) 22 The possible scores are exactly the odd numbers $-21, -19, \ldots, 19, 21$. To see that only odd numbers can occur, first consider the perfect score: 21 points. Now for every wrong answer, an even number must be subtracted from it, leaving an odd number. Conversely, all odd numbers between -21 and 21 are possible scores. Indeed, if none or exactly one question is answered incorrectly, the resulting scores are $21, 19, 17, 15, 13, 11$ and 9. Answering incorrectly question 6 and one of the first four questions results in scores $7, 5, 3$ and 1. Similarly, answering at most one question correctly, or answering correctly question 6 and one of the first four questions, result in scores $-21, -19, \ldots, -1$.

A3. B) $\frac{2}{3}$ Connect the center M to A, C and E. Also connect C and E. Hence the hexagon is split into six equal triangles, four of which form the kite.

A4. B) 37 After three rounds, every player has lost 1 chip, and the pot has gained 3. After $12 \cdot 3$ rounds, the players have 1, 2 and 3 chips left respectively and 36 chips are in the pot. In the next and final round, one more chip is added to the pot.

A5. A) 1 If two numbers have a 1 as their last digit, also their product ends with a 1. Since the last digit of $7^4 = 2401$ equals 1, the same holds for every power of 7^4. In particular, the last digit of $((((7^6)^5)^4)^3)^2 = 7^{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = (7^4)^{180}$ is equal to 1.

A6. B) 4 Let $a = (\sqrt{2} + 1)^7$ and $b = (\sqrt{2} - 1)^7$. Then the given expression is equal to
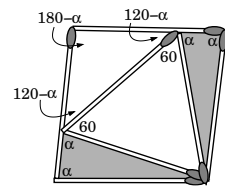
$$(a + b)^2 - (a - b)^2 = 4ab = 4(\sqrt{2} + 1)^7(\sqrt{2} - 1)^7 = 4((\sqrt{2} + 1)(\sqrt{2} - 1))^7 = 4 \cdot 1^7 = 4.$$

A7. B) 1467 After 9 miles, the second digit from the right is increased. After nine increments, so after $9 \cdot 9$ miles, the third digit from the right is increased. After $9 \cdot 9 \cdot 9 = 729$ miles the fourth digit from the right is increased. As the display shows 002010, the total distance traveled must be equal to $2 \cdot 729 + 1 \cdot 9 = 1467$ miles.

A8. E) 21 places to the right of Paul   Among 6 selected people, Paul is tallest. Therefore at least 5 people are shorter than Paul. James is shortest among 6 selected people, so at least 5 people are taller than James. James can therefore not be 21 places to the right of Paul, since then at least $5 + 1 + 20 + 1 + 5 = 32$ people would be standing in line.
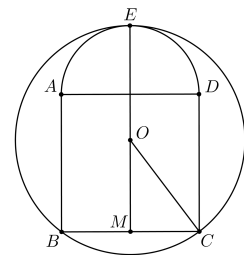
The other positions are indeed possible. Here we only show this for answer C: James and Paul staan naast elkaar. Number the people in increasing height from 1 to 30. Put numbers $1, 2, 3, 4$ and 10 in one row and put one of the numbers $5, 6, 7, 8$ en 9 in each of the other rows. The rest may be distributed freely over the remaining positions. The shortest people in the six rows are numbers $1, 5, 6, 7, 8$ and 9. Therefore Paul has nummer 9. James has nummer 10, since he is the tallest of his row and in every other row, there is someone with a number bigger than 10.

B1. 100   The outer four matches form a parallelogram. The four angles at the bases of the two isosceles triangles are therefore all equal. This angle, denoted by α equals 180 degrees minus the angle we are looking for. The inner three matches form an equilateral triangle (angles equal to 60 degrees). As the angles of any triangle add up to 180 degrees, we have that $(180 - \alpha) + 2(120 - \alpha) = 180$. It follows that $\alpha = 80$ and therefore the angle in the question measures 100 degrees.

B2. 4   The statement that dividing 2216 by a leaves a remainder of 29, is the same as saying that a divides $2216 - 29 = 2187$ and is larger than 29 (the remainder is always smaller than the divisor a). The divisors of $2187 = 3^7$ larger than 29 are $81, 243, 729$ en 2187. There are 4 choices for a.

B3. $\frac{5}{6}$ Let O be the center of the circumscribed circle, E the point where it touches the semicircle and let M be the midpoint of BC. Applying the Pythagorean theorem gives $|OC|^2 = |MC|^2 + |OM|^2$. Since $|OM| = |EM| - |OE| = \frac{3}{2} - |OC|$ this implies that $|OC|^2 = (\frac{1}{2})^2 + (\frac{3}{2} - |OC|)^2$. Solving this equation, we obtain $|OC| = \frac{5}{6}$.

B4. 5185   Number the rows of the array from top to bottom using 0 to 27 and the columns from left to right using 0 to 36. Consider the position in row r and column k. The red number written equals $1 + k + 37r$ and the green number equals $1 + r + 28k$. These two numbers are equal exactly when $36r = 27k$, that is, if and only if $4r = 3k$. As solutions we obtain for r the multiples of three: $0, 3, \ldots, 27$ and for k the matching multiples of four $0, 4, \ldots, 36$. The colored numbers in the corresponding ten positions are $1, 1 + 115, 1 + 2 \cdot 115, \ldots, 1 + 9 \cdot 115$. Adding these numbers we find the solution: $(1 + (1 + 9 \cdot 115)) \cdot 5 = 5185$.