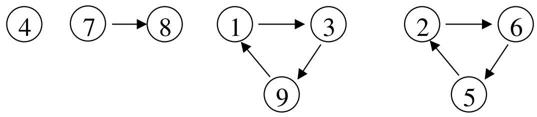
Second round
Dutch Mathematical Olympiad

Friday 13 March 2015

Solutions

B-problems

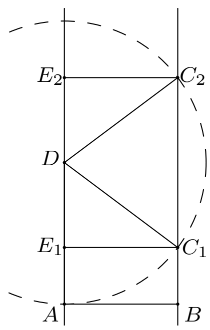
B1. 6 The multiples of 13 consisting of two digits are 13, 26, 39, 52, 65, 78, and 91. In a thirteenish number a 1 can only be followed by a 3, and a 2 can only be followed by a 6. The digit 4 cannot be followed by any digit. In the figure, each digit and its possible successors are shown.



For the first digit of a thirteenish number containing more than two digits, we can only pick from 1, 3, 9, 2, 6, and 5. Consequently all following digits are uniquely determined.

We now conclude that 13913, 26526, 39139, 52652, 65265, and 91391 are the thirteenish numbers of five digits. Hence, there are 6 of them.

B2. $6 \pm \sqrt{11}$ Consider line segment AB with length 5 and two lines perpendicular to it through the points A and B. Point D lies on the line through A at distance 6 from A, and point C lies on the line thorugh B on the same side of AB as point D. Because $|CD| = 6$, point C lies on the circle with centre D and radius 6. This gives two possibilities for the point C, which we denote by C_1 and C_2 (see the figure). Now take orthogonal projections of C_1 and C_2 on AD and call them E_1 and E_2, respectively. The Pythagorean theorem now yields



$$|DE_1|^2 = |DC_1|^2 - |C_1E_1|^2 = 6^2 - 5^2 = 11.$$

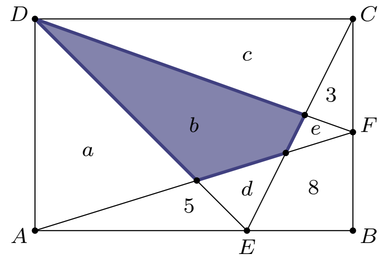
Hence, $|DE_1| = \sqrt{11}$ and $|BC_1| = |AE_1| = |AD| - |DE_1| = 6 - \sqrt{11}$. Similarly, we see that $|DE_2| = \sqrt{11}$ and therefore $|BC_2| = |AE_2| = |AD| + |DE_2| = 6 + \sqrt{11}$.
Consequently, the possible values of $|BC|$ are $6 - \sqrt{11}$ and $6 + \sqrt{11}$.

B3. 189 Let n be the number of people (including Berry). At first, everyone gets $\frac{756}{n}$ raspberries. Then, the three friends each give $\frac{1}{4} \cdot \frac{756}{n} = \frac{189}{n}$ raspberries back to Berry. From this it follows that 189 must be divisible by n.

The number of friends is at least 3, hence $n \geqslant 4$. We also have that $n \leqslant 8$. Indeed, if $n \geqslant 9$, then everyone would get less than $\frac{756}{9} = 84$ raspberries to start with, which implies that Berry eats less than $84 + 3 \times 21 = 147$ raspberries in total. This contradicts the fact that he ate more than 150 of them.

Out of the options $n = 4, 5, 6, 7, 8$, the only possibility is $n = 7$ because 189 is not divisible by 4, 5, 6 or 8. Hence, everyone gets $\frac{756}{7} = 108$ raspberries, after which the three friends each give $\frac{108}{4} = 27$ raspberries back. Hence, Berry ate $108 + 3 \times 27 = 189$ raspberries in total.

B4. 16 Denote the points and areas as in the figure. The area of triangle ADF is half the area of rectangle $ABCD$ and is equal to $a+b+e$. The area of triangle CDE is also half the area of rectangle $ABCD$ and is equal to $b+c+d$. If we add these two areas, we get the area of the whole rectangle. In other words, we have



$$(a+b+e)+(b+c+d)=a+b+c+d+e+3+5+8.$$

By subtracting $a+b+c+d+e$ of both sides of the equation, we find that $b=3+5+8=16$.

B5. 5 Look at the left figure below this paragraph. A well-placed one and a well-placed five can only be put in a square marked A because all other numbers in that same row/column have to be on the same side of the number. A well-placed two or four can only be put in a square marked B and a well-placed three can only be put in a square marked C.

A				A
	B		B	
		C		
	B		B	
A				A

A	5			A
5	**4**		B	
		C		
	B		B	
A				A

Because there cannot be two equal digits in a row or column, there cannot be more than two well-placed ones. Similarly, there are no more than two well-placed twos, fours or fives. If there is a well-placed four, then the two adjacent squares must contain fives (see the right figure). On three of the four squares marked A there cannot be a five anymore, because otherwise there would be two fives in a row or column. With two well-placed fours there cannot be any five on one of the four squares that are marked A. Therefore, the total number of well-placed fours and fives is not greater than two.

In a similar way the total number of well placed twos and ones cannot be greater than two. Hence, we may conclude that there can be at most $2+2+1=5$ well-placed numbers in total.

Now we shall show that it is possible to have an arrangement with 5 well-placed numbers. Though one example would suffice to complete the proof, we give two examples of squares with 5 well-places numbers.

1	3	2	4	**5**
2	4	1	5	3
4	5	**3**	1	2
3	1	5	2	4
5	2	4	3	**1**

1	5	2	4	3
5	**4**	1	3	2
2	1	**3**	5	4
4	3	5	**2**	1
3	2	4	1	**5**

C-problems

C1. a) Kees starts with three numbers $a < b < c$. The three sums are then ordered as follows: $a+b < a+c < b+c$. If these are evenly spread, then the difference $(b+c)-(a+c) = b-a$ equals $(a+c)-(a+b) = c-b$. This is exactly the condition for the three original numbers a, b and c to be evenly spread. Hence, Jan was right.

b) Jan can accomplish this by taking the four numbers $0, 1, 2$ and 4 to start with. The six results then are $0+1=1$, $0+2=2$, $1+2=3$, $0+4=4$, $1+4=5$, and $2+4=6$. These six numbers are evenly spread. $\square$

C2. In the following cases, a colouring meeting all the requirements exists.

- $m = n$ is even
  We colour the board as in a chessboard pattern. That is: in each row and column the squares are alternately black and white. This colouring meets all requirements.
- $n = 2m$
  We colour all the squares in the left half of the board white, and colour all the squares in the right half of the board black. This colouring meets all requirements.

Now we shall show that these are the only possible board sizes. Consider a coloured board that meets all requirements. Because each row has equally many black and white squares, the total number of squares in a row must be divisible by 2. Write $n = 2k$. Each row has exactly k white and k black squares. Now consider the left column. If all its squares are white, then the column has k white squares because of the second requirement. Hence, we have $m = k$ in this case. The same happens when all squares in the left column are black. If there are both black and white squares in the left column, then there must be exacly k white and k black squares because of the second requirement. Hence, we find $m = 2k = n$ in this case.

We conclude that for a pair (m, n) there exists a colouring if and only if $n = 2m$ or if $m = n$ and n is even. $\square$