Second round
Dutch Mathematical Olympiad

Friday 23 March 2012

Solutions

B-problems

B1. 32 We will reason through the addition from right to left. Either 2E = 12 or 2E = 2. The second case is excluded because then 2D would equal 1 or 11. Therefore E = 6.

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T W E E D E
  R O N D E +
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2 3 0 3 1 2
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Either D = 0 or D = 5. The second case is excluded because then E + N + 1 = 13 or N = 6 would hold, while 6 is already taken by E. So D = 0 and E + N = 13, which implies that N = 7.

From E + O + 1 = 10 it follows that O = 3. Hence W + R = 12 or W + R = 2. The second case is excluded: W + R $\geqslant 1 + 2 = 3$ because 0 is already taken. Hence, W + R = 12 (and T = 1).

The pair {W, R} must be one of the following: $\{3, 9\}$, $\{4, 8\}$, and $\{5, 7\}$. The first and last possibility are excluded because 3 and 7 are already taken. We conclude that W · R = $8 \cdot 4 = 32$.

B2. 70 We may put 23 camels in one pasture, and put $32, 33, 34, \dots, 70$ camels in the remaining 39 pastures, respectively. The last pasture is the one in the centre of Amsterdam. The total number of assigned camels indeed equals $23 + (32 + 33 + \cdots + 70) = 23 + 39 \cdot \frac{32+70}{2} = 2012$. Hence, a valid distribution having 70 camels in the pasture in the centre of Amsterdam exists.

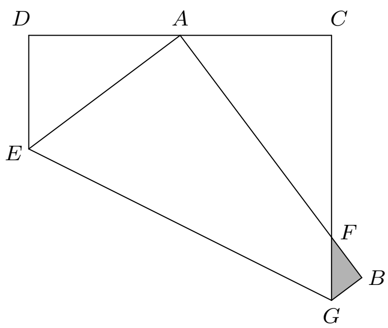
It is not possible to do it with less than 70. Indeed, suppose that at most 69 camels are put in the pasture in the centre of Amsterdam. Because the number of camels is different for each pasture, the second most populated pasture has at most 68 camels, the third most populated pasture has at most 67 camels, and so on. In total the pastures have at most $69 + 68 + \ldots + 30 = 40 \cdot \frac{30+69}{2} = 1980$ camels, leaving at least 32 unassigned camels. Therefore, no solution exists with fewer than 70 camels in the pasture in the centre of Amsterdam. The minimum is therefore 70.

B3. 5 First consider the case that Anne has stolen the king's gold. The last statement of both Bert and Chris is true in this case, and the last statement of both Anne and Dirk is false. Hence, Anne and Dirk are liars. Since both of Chris' statements are true, Chris is no liar. This implies that Bert is a liar, because his first statement was a lie. Now that we know who is a liar and who is not, it is easy to see that exactly five of the eight statements are true.
In the cases where Bert, Chris or Dirk is the thief, a similar reasoning holds. By the symmetry of the problem (cyclicly permuting the names 'Anne', 'Bert', 'Chris', and 'Dirk' does not change the problem), exactly five of the statements are true in each case.

B4. -178 Colour the 10,000 squares in a chess board pattern: the upper left square and bottom right square will be white and the squares in the upper right and bottom left will be black. Consider the $99 \cdot 99 = 9801$ 2×2-blocks. The 4901 blocks having a white square in the upper left corner are called the *white blocks* and the 4900 blocks having a black square in the upper left corner are called the *black blocks*. For each white block, add the four numbers it contains. Let W be the result of adding these 4901 outcomes (some squares are counted twice!). Let Z be the result when applying the same procedure to the 4900 black blocks. Since the number of white blocks is one more than the number of black blocks, we have $W - Z = 20$.

Now consider for each square how many times it is counted in total. Each of the four corner squares is counted only in one white block. Each of the other squares at the edge of the board are counted in exactly one white block and one black block. Each of the remaining squares is counted in exactly two white blocks and two black blocks. Considering the difference $W - Z$, only the four corner squares are counted, each exactly once. For the number x in the bottom right corner, we find that $x + 0 + 99 + 99 = W - Z = 20$, hence $x = -178$.

B5. $\frac{2}{3}$ Introduce points E, F, and G as in the figure. Suppose that $|DE| = x$. Then $|AE| = 8-x$, because the square has sides of length 8. Using the Pythagorean theorem, we get $(8-x)^2 = |AE|^2 = |DE|^2+|AD|^2 = x^2+16$. Solving this quadratic equation gives $x = 3$. Observe that $\angle CAF = 180° - \angle DAE - \angle EAB = 90° - \angle DAE = \angle DEA$. Also, $\angle ADE = 90° = \angle FCA$ holds. Therefore, triangles DEA and CAF are similar (AA). This implies that $\frac{1}{4}|AF| = \frac{|AF|}{|AC|} = \frac{|EA|}{|DE|} = \frac{5}{3}$, and hence $|AF| = \frac{20}{3}$. We also see that $\frac{1}{4}|CF| = \frac{|CF|}{|AC|} = \frac{|AD|}{|DE|} = \frac{4}{3}$ and so $|CF| = \frac{16}{3}$. It follows that $|BF| = 8 - |AF| = \frac{4}{3}$. Since $\angle CFA = \angle BFG$ (vertical angles), and $\angle ACF = \angle GBF = 90°$, triangles CFA and BFG are similar (AA). This implies that $\frac{3}{4}|BG| = \frac{|BG|}{|BF|} = \frac{|AC|}{|CF|} = \frac{4}{\frac{16}{3}} = \frac{3}{4}$ and therefore $|BG| = 1$. It follows that the area of the grey triangle equals $\frac{1}{2}\cdot|BG|\cdot|BF| = \frac{1}{2}\cdot 1\cdot\frac{4}{3} = \frac{2}{3}$.



C-problems

C1. (a) Applying the two rules, we can make the following sequence of cards: $12 \to 4 \to 9 \to 3 \to 7 \to 15 \to 31 \to 63 \to 21 \to 43 \to 87 \to 29$.

(b) We have seen in part (a) how to make a card with the number $3 = 2^2 - 1$. Iterating the first rule, we can sequentially construct cards with numbers $2 \cdot (2^2 - 1) + 1 = 2^3 - 1$, $2 \cdot (2^3 - 1) + 1 = 2^4 - 1$, and so on. In particular, we can make the number $2^{2012} - 1$.

(c) Applying rule 1 to any card produces a card with an *odd* number. Applying rule 2 to a card with an *odd* number (assuming it is a multiple of three), produces a card with an *odd* number. As soon as we apply rule one, the resulting card will only give rise to cards with an *odd* number. To get *even* numbers, we must therefore restrict to using rule 2 only (repeatedly). The only *even* numbers that can be made are therefore 12 and 4.

C2. Observe that $\angle ESB = \angle DSC$ (vertical angles). Since triangles BES and SDC are isosceles, $\angle EBS = \angle ESB = \angle DSC = \angle DCS$. Hence triangles BES and SDC are similar (AA). In particular, $|BS| = \frac{|BS|}{|BE|} = \frac{|SC|}{|SD|} = \frac{1}{2}|SC| = |SM|$. Therefore, triangle BSM is isosceles and $\angle SBM = \angle SMB = \angle TMC$ (vertical angles). Using that the angles of a triangle sum to 180°, we find that $\angle TMC = 180° - \angle MTC - \angle TCM$ and hence also $\angle ATB = 180° - \angle MTC = \angle TMC + \angle TCM$. We had already shown that $\angle TMC = \angle SBM$ and $\angle TCM = \angle DCS = \angle ABS$. We therefore see that $\angle ATB = \angle SBM + \angle ABS = \angle ABT$. It follows that triangle BAT is isosceles with top A, and therefore $|AB| = |AT|$ holds.