Final round
Dutch Mathematical Olympiad

Friday 12 September 2014

Solutions

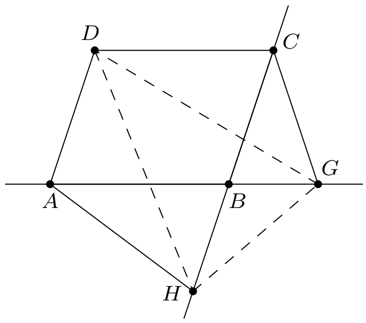
1. Suppose that (a,b,c) is a solution. From $a \leqslant b \leqslant c$ it follows that $abc = 2(a+b+c) \leqslant 6c$. Dividing by c yields $ab \leqslant 6$. We see that $a = 1$ or $a = 2$, because from $a \geqslant 3$ it would follow that $ab \geqslant a^2 \geqslant 9$.

We first consider the case $a = 2$.
From $ab \leqslant 6$ it follows that $b = 2$ or $b = 3$. In the former case, the equation $abc = 2(a+b+c)$ yields $4c = 8 + 2c$ and hence $c = 4$. It is easy to check that the triple $(2,2,4)$ we got is indeed a solution. In the latter case, we have $6c = 10 + 2c$, hence $c = \frac{5}{2}$. Because c has to be an integer, this does not give rise to a solution.

Now we consider the case $a = 1$.
We get that $bc = 2(1+b+c)$. We can rewrite this equation to obtain $(b-2)(c-2) = 6$. Remark that $b-2$ cannot be negative (and hence also $c-2$ cannot be negative). Otherwise, we would have $b = 1$, yielding $(1-2)(c-2) = 6$, from which it would follow that $c = -4$. However, c has to be positive.
There are only two ways to write 6 as a product of two non-negative integers, namely $6 = 1 \times 6$ and $6 = 2 \times 3$. This gives rise to two possibilities: $b-2 = 1$ and $c-2 = 6$, or $b-2 = 2$ and $c-2 = 3$. It is easy to check that the corresponding triples $(1,3,8)$ and $(1,4,5)$ are indeed solutions.

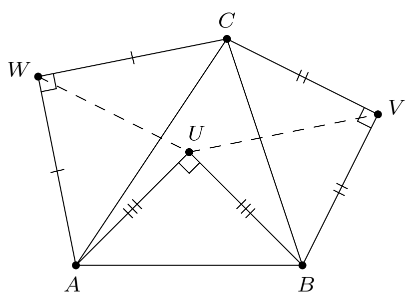
Thus, the only solutions are $(2,2,4)$, $(1,3,8)$, and $(1,4,5)$. □

2. Version for klas 5 & klas 4 en lager



We know that $\angle ABH = \angle CBG$, because these are opposite angles. Because triangles ABH and CBG are isosceles, we have $\angle AHB = \angle ABH$ and $\angle CBG = \angle CGB$. Triangles ABH and CBG are similar (AA) and hence we have $\angle BAH = \angle BCG$. Because $ABCD$ is a parallelogram, we have $\angle DAB = \angle DCB$ and hence $\angle DAH = \angle DAB + \angle BAH = \angle DCB + \angle BCG = \angle DCG$ holds. Because $ABCD$ is a parallelogram, we have $|CD| = |AB| = |AH|$ and $|AD| = |BC| = |CG|$. Therefore, triangles DAH and GCD are congruent (SAS) and we have $|DH| = |DG|$. In other words, triangle DGH is isosceles. □

2. Version for klas 6



Because triangle AUB is isosceles with top angle $\angle AUB = 90°$, we have $\angle UAB = 45°$. In the same way, we have $\angle CAW = 45°$. Combining these two equalities, we find $\angle WAU = 45° + \angle CAU = \angle CAB$. By the Pythagorean theorem, we find $2|AW|^2 = |AW|^2 + |CW|^2 = |AC|^2$ and hence $|AW| = \frac{1}{2}\sqrt{2} \cdot |AC|$. In the same way we find $|AU| = \frac{1}{2}\sqrt{2} \cdot |AB|$. Hence, triangles WAU and CAB are similar (SAS) with magnification factor $\frac{|AW|}{|AC|} = \frac{1}{2}\sqrt{2} = \frac{|AU|}{|AB|}$. In particular, we find $|WU| = \frac{1}{2}\sqrt{2} \cdot |BC| = |CV|$.

In the same way, we see that triangles VBU and CBA are similar and that $|VU| = \frac{1}{2}\sqrt{2} \cdot |AC| = |CW|$. It follows that in quadrilateral $UVCW$ the opposite sides have equal lengths, hence $UVCW$ is a parallelogram. □

3. a) Suppose that the number of teams is 6. We shall derive a contradiction.

First remark that the number of games equals $\frac{6\times 5}{2} = 15$. Hence, the total number of points also equals 15.

Let team A be the (only) team with the lowest score. Team A has *at most* 1 point, because if team A had 2 or more points, then each of the other five teams would have at least 3 points, giving a total number of points that is at least $2+3+3+3+3+3 = 17$. Each team on the second last place in the ranking has lost to team A, because this is the only team with a lower score. Hence, team A also has *at least* 1 point. We deduce that A has exactly 1 point and that there is exactly one team, say team B, in the second last place in the ranking.

Team B has at least 2 points and the remaining four teams, teams C, D, E and F, each have at least 3 points. The six teams together have at least $1+2+3+3+3+3 = 15$ points. If team B had more than 2 points, or if any of the teams C through F had more than 3 points, then the total number of points would be greater than 15, which is impossible. Hence, team B has exactly 2 points and teams C through F each have exactly 3 points. The four teams C through F each lost to a team having a lower score (team A or team B). Hence, together, team A and team B must have won at least 4 games. This contradicts the fact that together they have only $1+2 = 3$ points.

□

b) In the table below there is a possible outcome for 7 teams called A through G. In the row corresponding to a team, crosses indicate wins against other teams. Row 2, for example, indicates that team B won against teams C and D and obtained a total score of 2 points. Each team (except A) has indeed lost exactly one match against a team with a lower score. These matches are indicated in bold.

	A	B	C	D	E	F	G	Score
A	-	**X**						1
B		-	**X**	**X**				2
C	X		-		**X**		**X**	3
D	X		X	-		**X**		3
E	X	X		X	-	X		4
F	X	X	X			-	X	4
G	X	X		X	X		-	4

□

4. a) Without loss of generality, we may assume that $a < b < c$. The integers a and c are not divisible by p because that would imply that $ac+1$ is a multiple of p plus 1, hence not divisible by p. Since $bc+1$ and $ac+1$ are both divisible by p, their difference $(bc+1)-(ac+1) = (b-a)c$ is divisible by p as well. Hence, since c is not divisible by p, it must be the case that $b-a$ is divisible by p. Similarly, $(ac+1)-(ab+1) = a(c-b)$ is divisible by p and since a is not divisible by p, this implies that $c-b$ is divisible by p.
Thus, we find that $b = a + (b-a) \geqslant a + p$ and $c = b + (c-b) \geqslant a + 2p$.

We have $a \geqslant 2$. Indeed, suppose that $a = 1$. Then, both integers $b+1 = ab+1$ and $b-1 = b-a$ are divisible by p, which implies that their differnce $(b+1)-(b-1) = 2$ is divisible by p as well. However, p is an odd prime and can therefore not divide 2.

Using $a \geqslant 2$, $b \geqslant a+p$, and $c \geqslant a+2p$, we conclude that

$$\frac{a+b+c}{3} \geqslant \frac{a+(a+p)+(a+2p)}{3} = p + a \geqslant p + 2.$$

□

Remark. *The above proof uses that fact that p is a prime to conclude that p divides $b-a$ or c given that it divides the product $(b-a)c$. It turns out, that in the problem statement we can relax the requirement that p is a prime and only demand that p is an integer larger than 2. The problem statement remains valid, as follows from the following sketch of an alternative proof.*

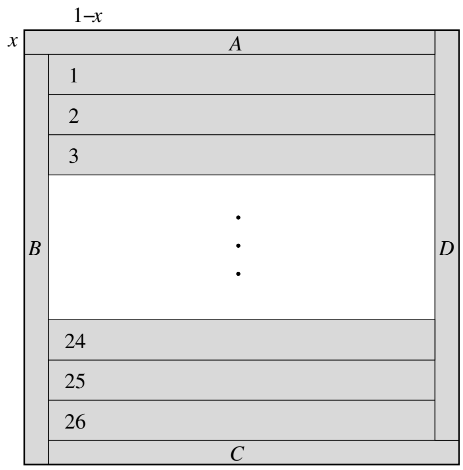
Again, we may assume that $a < b < c$. Observe that $a(bc+1) = abc + a$, $b(ac+1) = abc + b$, and $c(ab+1) = abc + c$ are different multiples of p. Hence, the differences $(abc+b) - (abc+a) = b - a$ and $(abc+c) - (abc+b) = c - b$ are multiples of p as well. Again, we can conclude that $b \geqslant a + p$ and $c \geqslant b + p \geqslant a + 2p$. The remainder of the proof is the same as in the first proof.

b) Again, we may assume that $a < b < c$. In part a) we have seen that $\frac{a+b+c}{3} \geqslant \frac{a+(a+p)+(a+2p)}{3} = p + a \geqslant p + 2$. We can only have $\frac{a+b+c}{3} = p + 2$ if $b = a + p$, $c = a + 2p$, and $a = 2$. Since $ab + 1 = 2(2 + p) + 1 = 2p + 5$ must be divisible by p, it follows that 5 is divisible by p. We conclude that $p = 5$, $b = 7$, and $c = 12$. The quadruple $(p, a, b, c) = (5, 2, 7, 12)$ is indeed a Leiden quadruple, because $ab + 1 = 15$, $ac + 1 = 25$, and $bc + 1 = 85$ are all divisible by p.

We conclude that $p = 5$ is the only number for which there is a Leiden quadruple (p, a, b, c) that satisfies $\frac{a+b+c}{3} = p + 2$. □

5. a) Consider a rectangle with sides of length $a \leqslant b$ inside the square. Since $b \leqslant 1$ and $2a+2b = \frac{5}{2}$ hold, we see that $a \geqslant \frac{1}{4}$. The area of the rectangle equals ab and is therefore at least $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$. Hence, we can have no more than 16 rectangles inside the square without creating overlaps. □

b) A solution is sketched in the figure below. The four outer rectangles, A through D, are equal with the shorter side having length x, and the longer side having length $1 - x$. Together they leave uncovered a square area with sides of length $1 - 2x$. This area is then tiled by 26 equal rectangles. These have sides of length $1 - 2x$ and $\frac{1-2x}{26}$, and therefore have a circumference of $\frac{54}{26}(1 - 2x)$. To obtain a circumference of length 2, we take $x = \frac{1}{54}$.



□