Final round Dutch Mathematical Olympiad

Friday 16 September 2011

Solutions

1. Since a and b play the same role in the equation $a! + b! = 2^n$, we will assume for simplicity that $a \leqslant b$. The solutions for which $a > b$ are found by interchanging a and b. We will consider the possible values of a.

Case $a \geqslant 3$: Since $3 \leqslant a \leqslant b$, both $a!$ and $b!$ are divisible by 3. Hence $a! + b!$ is divisible by 3 as well. Because 2^n is not divisible by 3 for any value of n, we find no solutions in this case.

Case $a = 1$: The number b must satisfy $b! = 2^n - 1$. This implies that $b!$ is *odd*, because 2^n is *even* (recall that $n \geqslant 1$). Since $b!$ is divisible by 2 for all $b \geqslant 2$, we must have $b = 1$. We find that $1! = 2^n - 1$, which implies that $n = 1$. The single solution in the case is therefore $(a, b, n) = (1, 1, 1)$.

Case $a = 2$: There are no solutions for $b \geqslant 4$. Indeed, since $b!$ would then be divisible by 4, $2^n = b! + 2$ would *not* be divisible by 4, which implies that $2^n = 2$. However, this contradicts the fact that $2^n = b! + 2 \geqslant 24 + 2$.

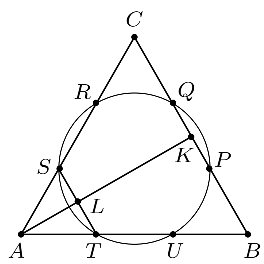
For $b = 2$, we find $2^n = 2 + 2 = 4$. Hence $n = 2$ and $(a, b, n) = (2, 2, 2)$ is the only solution.

For $b = 3$, we find $2^n = 2 + 6 = 8$. Hence $n = 3$ and $(a, b, n) = (2, 3, 3)$ is the only solution.

By interchanging a and b, we obtain the additional solution $(a, b, n) = (3, 2, 3)$.

In all, there are four solutions: $(1, 1, 1)$, $(2, 2, 2)$, $(2, 3, 3)$ and $(3, 2, 3)$. □

2. Denote by K the midpoint of PQ. Then K is also the midpoint of BC, and AK is a median of triangle ABC. We denote by L the intersection of AK and ST.



Triangles AST and ACB are similar (sas), because $\angle CAB = \angle SAT$ and $\frac{|CA|}{|SA|} = 3 = \frac{|BA|}{|TA|}$. This implies that ST and CB are parallel lines (equal corresponding angles).

Triangles ASL and ACK are similar (aa), because $\angle SAL = \angle CAK$ and $\angle LSA = \angle TSA = \angle BCA = \angle KCA$. Hence $\frac{|CK|}{|SL|} = \frac{|CA|}{|SA|} = 3$. This implies that L is the midpoint of ST, because $\frac{|SL|}{|ST|} = \frac{3 \cdot |SL|}{3 \cdot |ST|} = \frac{|CK|}{|CB|} = \frac{1}{2}$.

Consider the center M of the circle through P, Q, R, S, T and U. It is incident to both the perpendicular bisector of PQ, and that of ST. However, since PQ and ST are parallel, the two perpendicular bisectors must coincide: they are the same line. This line is incident to L and K, and is therefore equal to line AK, which shows that $AK \perp BC$.

It follows that $|AC| = |AB|$, because AK is the perpendicular bisector of BC.

In a similar fashion, one can show that $|AC| = |BC|$, concluding the proof that triangle ABC is equilateral. □

3. In all, 15 matches are played. In each match, the two teams together earn 2 or 3 points. The sum of the final scores is therefore an integer between $15 \cdot 2 = 30$ (all matches end in a draw) and $15 \cdot 3 = 45$ (no match is a draw).

On the other hand, the sum of the six scores equals $a + (a + 1) + \cdots + (a + 5) = 15 + 6a$. Hence $30 \leqslant 15 + 6a \leqslant 45$, which shows that $3 \leqslant a \leqslant 5$. We will prove that $a = 4$ is the only possibility.

First consider the case $a = 5$. The sum of the scores equals $15 + 30 = 45$, so no match ends in a draw. Because in every match the teams earn either 0 or 3 points, every team's score is divisible by 3. Therefore, the scores cannot be six consecutive numbers.

Next, consider the case $a = 3$. The scores sum up to $3+4+5+6+7+8 = 33$. The two teams scoring 6 and 7 points must both have won at least one out of the five matches they played.

The team scoring 8 points must have won at least two matches, because $3+1+1+1+1 = 7 < 8$. Hence at least 4 matches did not end in a draw, which implies that the sum of the scores is at least $4 \cdot 3 + 11 \cdot 2 = 34$. But we have already see that this sum equals 33, a contradiction.

Finally, we will show that $a = 4$ is possible. The table depicts a possible outcome in which teams A to F have scores 4 to 9. The rightmost column shows the total scores of the six teams. ☐

	A	B	C	D	E	F	
A	-	3	1	0	0	0	4
B	0	-	1	0	3	1	5
C	1	1	-	3	0	1	6
D	3	3	0	-	1	0	7
E	3	0	3	1	-	1	8
F	3	1	1	3	1	-	9

4. For convenience, write $y = \sqrt{a}$ and $z = \sqrt{b}$. The equations transform to

$$y^3 + z^3 = 134 \qquad \text{and} \qquad y^2z + yz^2 = 126.$$

Combining these two equations in a handy way, we find

$$(y+z)^3 = (y^3+z^3) + 3(y^2z + yz^2) = 134 + 3 \cdot 126 = 512 = 8^3.$$

This immediately implies that $y + z = 8$.

Rewrite the first equation as follows: $(y+z)yz = y^2z + yz^2 = 126$. Since $y+z = 8$, we see that $yz = \frac{126}{8} = \frac{63}{4}$.

From $y+z = 8$ and $yz = \frac{63}{4}$, we can determine y and z by solving a quadratic equation: y and z are precisely the roots of the equation $x^2 - 8x + \frac{63}{4} = 0$. The two solutions are $\frac{8 \pm \sqrt{64 - 4 \cdot \frac{63}{4}}}{2}$, that is $\frac{9}{2}$ and $\frac{7}{2}$.

Since $a > b$, also $y > z$ holds. Hence $y = \frac{9}{2}$ and $z = \frac{7}{2}$. We therefore find that $(a,b) = (\frac{81}{4}, \frac{49}{4})$.

Because $(a,b) = (\frac{81}{4}, \frac{49}{4})$ satisfies the given equations, as required, we conclude that this is the (only) solution. ☐

5. We are give that 1 is white. Hence 0 is black, because otherwise $1 = 1 - 0$ and $1 = 1 + 0$ would have different colours. The number 2 is white, because $0 = 1 - 1$ (black) and $2 = 1 + 1$ have different colours.

By induction on k, we show that the following claim holds for every $k \geqslant 0$:

$3k$ is black, $3k+1$ and $3k+2$ are white.

We have just shown the base case $k = 0$. Assume that the claim holds true for $k = \ell$.

Since 1 is white, and $3\ell+2$ is white by the induction hypothesis, the numbers $(3\ell+2)-1 = 3\ell+1$ and $(3\ell+2)+1 = 3(\ell+1)$ have different colours. As $3\ell+1$ is white by the induction hypothesis, $3(\ell+1)$ must be black.

Since 2 and $3\ell+2$ are both white, the numbers $(3\ell+2)+2 = 3(\ell+1)+1$ and $(3\ell+2)-2 = 3\ell$ must have different colours. As 3ℓ is black by the induction hypothesis, $3(\ell+1)+1$ must be white.

Since $3(\ell+1)+1$ and 1 are both white, the numbers $3(\ell+1)+1+1 = 3(\ell+1)+2$ and $3(\ell+1)$ have different colours. We already know that $3(\ell+1)$ is black, so $3(\ell+1)+2$ must be white.

This proves the claim for $k = \ell + 1$.

Because $2011 = 3 \cdot 670 + 1$, this shows that 2011 is white. ☐