Final round
Dutch Mathematical Olympiad

Friday 16 September 2011
Technical University Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. A description of your solution method and clear argumentation are just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem. Good luck!

1. Determine all triples of positive integers (a, b, n) that satisfy the following equation:

$$a! + b! = 2^n.$$

Notation: $k! = 1 \times 2 \times \cdots \times k$, for example: $1! = 1$, and $4! = 1 \times 2 \times 3 \times 4 = 24$.

2. Let ABC be a triangle. Points P and Q lie on side BC and satisfy $|BP| = |PQ| = |QC| = \frac{1}{3}|BC|$. Points R and S lie on side CA and satisfy $|CR| = |RS| = |SA| = \frac{1}{3}|CA|$. Finally, points T and U lie on side AB and satisfy $|AT| = |TU| = |UB| = \frac{1}{3}|AB|$. Points P, Q, R, S, T and U turn out to lie on a common circle.
Prove that ABC is an equilateral triangle.

3. In a tournament among six teams, every team plays against each other team exactly once. When a team wins, it receives 3 points and the losing team receives 0 points. If the game is a draw, the two teams receive 1 point each.
Can the final scores of the six teams be six consecutive numbers $a, a+1, \ldots, a+5$? If so, determine all values of a for which this is possible.

4. Determine all pairs of positive real numbers (a, b) with $a > b$ that satisfy the following equations:

$$a\sqrt{a} + b\sqrt{b} = 134 \qquad \text{and} \qquad a\sqrt{b} + b\sqrt{a} = 126.$$

5. The number devil has coloured the integer numbers: every integer is coloured either black or white. The number 1 is coloured white. For every two white numbers a and b (a and b are allowed to be equal) the numbers $a - b$ and $a + b$ have different colours.
Prove that 2011 is coloured white.