First Round
Dutch Mathematical Olympiad

19 January – 29 January 2015

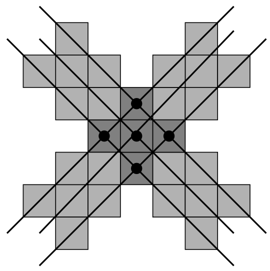
Solutions

A1. A) 5 Adding the circumferences of the two rectangles, we obtain the circumference of the square plus twice the length of the line segment cutting the square. This line segment and the sides of the square have the same length. Therefore, the sum of the circumferences of the two rectangles equals six times the side length of the square. It follows that the square has sides of length $\frac{30}{6} = 5$ centimetres.

A2. E) Erik Aad, Bas, and Carl cannot all have spoken the truth, because this would imply that both Dave and Erik lied. Since Aad, Bas, or Carl lied, both Dave and Erik must have told the truth. Aad and Bas cannot both have told the truth, because then Dave would have lied, contradicting our previous conclusion. Similarly, Bas and Carl cannot both have told the truth, because then Erik would have lied.

Since the liar must be either Aad or Bas, and also either Bas or Carl, we conclude that it must be Bas who is the liar. It follows that Aad was the first to arrive and that Carl arrived third. Erik arrived in between Bas and Carl. This is possible only if Bas arrived last and Erik arrived fourth.

A3. A) 12081 The diagonals running from top-right to bottom-left, will be called *anti-diagonals* from now on. The main diagonal consists of 2015 grey squares and the two adjacent diagonals consist of 2014 grey squares each. A similar statement holds for the anti-diagonals.



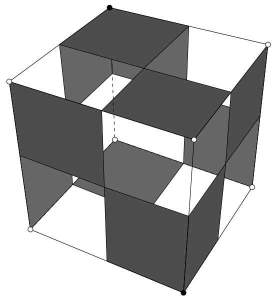
The main diagonal intersects the main anti-diagonal in a square, but does not intersect the adjacent anti-diagonals. Each of the two diagonals adjacent to the main diagonal intersects each of the two anti-diagonals adjacent to the main anti-diagonal, but not the main anti-diagonal itself.

This implies that adding the number of squares on the three diagonals and three anti-diagonals, there are $1 + 2 \times 2 = 5$ squares which are counted twice. Hence the total number of grey squares equals $2 \times 2015 + 4 \times 2014 - 5 = 12081$.

A4. E) 22475 Let n be the average of the two numbers. Then one number equals $n - 5$ and the other equals $n + 5$. Their product is $(n-5)(n+5) = n^2 - 25$. We will therefore add 25 to each of the five suggested answers to see which gives a perfect square. We obtain: 22423, 22445, 22467, 22478, and 22500. Clearly, $22500 = 150^2$ is a perfect square, so option E is correct.

To check that the other options are wrong, we calculate $149^2 = (150-1)^2 = 22500 - 300 + 1 = 22201$. Since $149^2 < 22423$, the options A through D are indeed incorrect.

A5. C) 2 After dividing the faces of the cube, there are 24 squares: 12 white ones and 12 black ones. The number of dark vertices cannot be zero, because that would imply a total of no more than $8 \times 1 = 8$ black squares (one for each corner). The number of dark vertices cannot be one, because that would imply a total of no more than $3 + 7 \times 1 = 10$ black squares. A solution with exactly two dark vertices is shown in the figure. It follows that the minimum number of dark vertices is 2.



A6. B) 2 Suppose that we are summing n numbers. There are two cases to consider.

n **is odd** In this case, there is a middle number, say k. The sum of the n numbers then equals $n \times k = 100$. Since n must be an odd divisor of 100, it must be equal to 5 or 25. In the first case, we find $k = \frac{100}{5} = 20$, with corresponding solution $18 + 19 + 20 + 21 + 22 = 100$. In the second case, we find $k = 4$. But this yields no solution, because the smallest of the 25 numbers would then equal $4 - 12$, which is negative.

n **is even** Write $n = 2m$. The two middle numbers add up to an odd number, say k. The summation now consists of m pairs of numbers, each pair adding up to k. Hence, $100 = k \times m$. Since k is an odd divisor of 100, it follows that $k = 5$ or $k = 25$. In the first case, $m = 20$ and the middle two numbers are 2 and 3. This does not yield a solution, because otherwise the smallest number would be $3 - 20$, which is negative. In the second case, $m = 4$, with corresponding solution $100 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16$.

A7. C) 18 The node in the centre cannot be coloured red, because otherwise we would create a horizontal or vertical line containing three red nodes. Both the horizontal and the vertical line need two red nodes. There are $\frac{4 \times 3}{2} = 6$ ways of colouring two nodes on the horizontal line. In two cases, the two coloured nodes lie on a common circle. Then, the vertical line has only two nodes left that may be couloured red. This can be done in only one way. In the other four cases, the two red nodes lie on different circles. On the vertical line, we need to colour one node from each circle. This can be done in $2 \times 2 = 4$ ways. In total, Jaap has $2 \times 1 + 4 \times 4 = 18$ ways of colouring the nodes.

A8. D) 1013 The number of newly grown branches doubles every day. Hence, on days $1, 2, 3, \ldots$ an additional $1, 2, 4, 8, \ldots$ branches are grown. In general, 2^{n-1} new branches are grown on day n.

The total numbers of branches on the consecutive days are therefore 1, $1 + 2 = 3$, $1 + 2 + 4 = 7$, $1 + 2 + 4 + 8 = 15$, and so on. On day n, the tree has a total of $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$ branches.

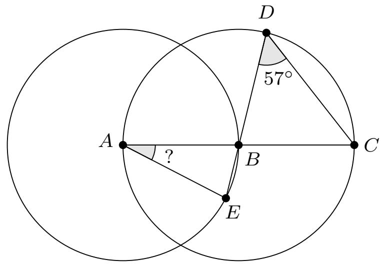
Each day, the number of leaves increases by the number of branches on the day before. Therefore, on the consecutive days there are 0, $0 + 1$, $0 + 1 + 3$, $0 + 1 + 3 + 7, \ldots$ leaves. On day n, the number of leaves equals

$$\begin{aligned}(2^0 - 1) + (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{n-1} - 1) &= (2^0 + 2^1 + \cdots + 2^{n-1}) - n \\ &= (2^n - 1) - n.\end{aligned}$$

It follows that the total number of leaves at the end of day 10 equals $2^{10} - 11 = 1013$.

B1. 35 For any three consecutive numbers a, b, c from Julia's sequence, we know that $c = 2b - a$. In other words: $c - b = b - a$. This means that the difference between pairs of consecutive numbers is constant throughout the sequence. Let this difference be d. Hence, if x is the first number from the sequence, then the sequence reads $x,\ x+d,\ x+2d,\ x+3d, \ldots$. The second number from the sequence equals $55 = x + d$, and the 100th number equals $2015 = x + 99d$. We see that $2015 - 55 = 98d$, and hence $d = \frac{1960}{98} = 20$. Using this, we see that $55 = x + 20$. We conclude that the first number from Julia's sequence is $x = 35$.

B2. 48° The angle at D equals the angle at C, because CBD is an isosceles triangle (with apex B). Angle DBC equals $180^\circ - 2 \times 57^\circ = 66^\circ$. Hence, also angle ABE is 66 degrees. Since triangle BAE is isosceles (apex A), the angle at E is 66 degrees as well. The angle at A is therefore equal to $180^\circ - 2 \times 66^\circ = 48^\circ$.



B3. 70 Let $n = abcd$ be a four digit number, where a is nonzero. We write $2n = efghi$, where the first digit, e, may be equal to 0. First observe the following.

- Digit i is *even*. Indeed it is equal to $2d$ or to $2d - 10$.
- Digit h is *even* if $d \leqslant 4$, and *odd* if $d \geqslant 5$.
- Digit g is *even* if $c \leqslant 4$, and *odd* if $c \geqslant 5$.
- Digit f is *even* if $b \leqslant 4$, and *odd* if $b \geqslant 5$.
- Digit e is *even* if $a \leqslant 4$, and *odd* if $a \geqslant 5$.

The number $2n$ is therefore alternating if and only if $d \geqslant 5$, $c \leqslant 4$, $b \geqslant 5$, and $a \leqslant 4$. In order to count the number of super alternating numbers, we consider two cases.

1. Digits a, c are *even* and b, d are *odd*. We can choose a from $2, 4$ (as 0 is not allowed), choose b from $5, 7, 9$, choose c from $0, 2, 4$, and choose d from $5, 7, 9$. This yields a total of $2 \times 3 \times 3 \times 3 = 54$ possibilities.

2. Digits b, d are *even* and a, c are *odd*. We can choose a from $1, 3$, choose b from $6, 8$, choose c from $1, 3$, and choose d from $6, 8$. This yields $2 \times 2 \times 2 \times 2 = 16$ possibilities.

The total number of super alternating numbers is therefore $54 + 16 = 70$.

B4. 66 For convenience, we number the students in the order in which they descend the mountain. Student 20, who is last to go down, has seen all numbers from 1 to 20 exactly once, receiving a card bearing 1 in the last round. In round 19, student 20 must have received a card bearing 2 (as there were only two cards that round). In round 18, he/she must have received a card bearing 3, and so on, until the first round, in which he/she received a card with number 20.

Now we consider student 19, who received a 1 in round 19. In round 18, student 19 must have received a 2 (the card bearing 3 was taken by student 20). In round 17, he/she must have received a 3 (student 20 got the card bearing 4, and numbers 1 and 2 are ruled out). This way, we see that student 19 received cards bearing numbers $1, 2, 3, \ldots, 19$ in rounds $19, 18, 17, \ldots, 1$, respectively.

Continuing this line of argumentation, we see that student n received cards with numbers $1, 2, 3, \ldots, n$ in rounds $n, n-1, n-2, \ldots, 1$, respectively. Since Sara got a card with 11 in round 1, she must be student number 11. The sum of the numbers on her cards is therefore $11 + 10 + 9 + \cdots = 2 + 1 = 66$.