First round
Dutch Mathematical Olympiad

20 January – 30 January 2014

Solutions

A1. B) 2 Suppose that we colour B2 black. Then the surrounding 8 cells cannot be coloured black. Indeed, the cells above, below, to the left, and to the right of B2 are in the same row or column as B2, while the other four cells are diagonally adjacent to B2. This way, only row 4 and column D remain and in each we can colour only one cell black. In total we can colour no more than 3 cells black. We conclude that B2 cannot be coloured black.

Similarly, we may deduce that cells B3, C2, and C3 cannot be coloured black. It follows that in row 2, we can only colour A2 or D2 black. If we colour A2 black, then in row 3 cell D3 must be coloured black because A3 is in the same column as A2. In rows 1 and 4 we now have no choice but to colour cells C1 and B4 black. This gives us one solution.

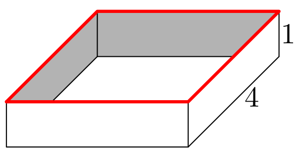
If instead of A2 we colour cell D2 black, then we find a solution where cells D2, A3, B1, and C4 are coloured black. In total, we have two ways of choosing the black squares.

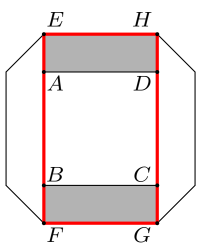
A2. D) $\frac{2}{5}$ From the given data, we deduce that $\frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$ of the carp are yellow females. Since half the carp are female, we find that $\frac{1}{2} - \frac{3}{10} = \frac{1}{5}$ of the carp are red females. Finally, using the fact that three fifths of the carp are red, we see that $\frac{3}{5} - \frac{1}{5} = \frac{2}{5}$ of the carp are red males.

A3. C) 3 and 5 The journey that visits the pads in the order 3, 6, 1, 4, 7, 2, and 5 (or in the opposite order), shows that pads 3 and 5 are the starting pad of a possible journey. We will see that these are the only possible starting points.

We say that two pads are *neighbours* if the frog can jump from one pad to the other (and hence also the other way around). Every intermediate pad in the frog's journey must have at least two neighbours: the pad the frog came from and the pad it will go next. Since pads 3 and 5 have only one neighbour (pad 6 and pad 2 respectively), these must be the starting point and end point of the frog's journey. The other pads must therefore be intermediate pads.

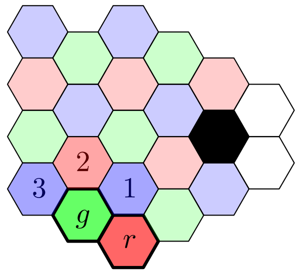
A4. B) 3 Consider the top rim of the paper ring (indicated in red). The rim has length $4 \times 4 = 16$. In the folded state, the rim becomes rectangle $EFGH$. Since $|AE| = |BF| = |CG| = |DH| = 1$, we find that $|AB| + |FG| + |CD| + |EH| = 16 - 4 = 12$. These four lengths equal the length of the sides of square $ABCD$. It follows that the square has sides of length $\frac{12}{4} = 3$.





A5. E) 45 Let n be the number of runners. The number of runners that finished before Tom equals $\frac{1}{2}(n-1)$ (half of all runners besides Tom). The number of runners that finished before Jerry equals $\frac{3}{4}(n-1)$. Since exactly 10 runners finished between Tom and Jerry, it follows that $\frac{3}{4}(n-1) - \frac{1}{2}(n-1)$ equals 11 (Tom and the 10 runners between Tom and Jerry). We find that $\frac{1}{4}(n-1) = 11$, hence $n = 4 \times 11 + 1 = 45$. There were 45 runners participating in the race.

A6. D) 18 We start by colouring the two indicated tiles at the bottom. This can be done in six ways: there are three options for the first tile and for each option there are two possible colours for the second tile. In the figure, the colours red (r) and green (g) are chosen.



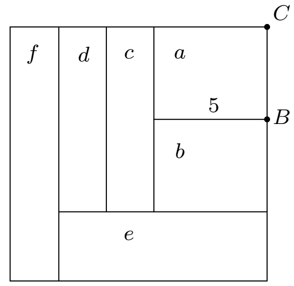
Now that these two tiles are coloured, the colours of most of the other tiles are determined as well. Tile number 1 can only be blue. Now tile number 2 must be red and therefore tile number 3 must be blue. In this way the colours of all tiles, except the two on the right (white in the figure), are fixed. For these last two tiles, there are three possible colourings. The upper and lower tile can be coloured either green and red, or blue and red, or blue and green.

Since each of the six allowed colourings of the first two tiles can be extended in three ways to a complete colouring, we find a total of $6 \times 3 = 18$ possible colourings.

A7. C) $\frac{25}{8}$ Observe that AMB is a straight angle. This implies that $\angle AMD + \angle DMC + \angle CMB = 180°$. Since triangles AMD and BMC are equal (three equal sides), we see that $\angle CMB = \angle MDA$. Hence $\angle DMC = 180° - \angle AMD - \angle MDA = \angle DAM$, because the angles of triangle AMD sum to 180 degrees. It follows that DMC and DAM are isosceles triangles with equal apex angles. Hence these two triangles are equal up to scaling. This means that $\frac{|CD|}{|DM|} = \frac{|DM|}{|AD|}$. Therefore, the length of CD equals $\frac{5}{8} \cdot 5 = \frac{25}{8}$.

A8. B) 4 From the mentioned point, it takes the same time to go 42% of the distance upstream and to go 58% of the distance downstream. This means that the boat is $\frac{58}{42}$ times as fast going downstream as going upstream. If the water flows at a speed of v kilometres per hour, then we find $\frac{25+v}{25-v} = \frac{58}{42}$. Hence $58 \cdot (25 - v) = 42 \cdot (25 + v)$, or $1450 - 58v = 1050 + 42v$. We find $400 = 100v$, hence $v = 4$.

B1. $\frac{24}{5}$ The six rectangles have equal areas. Rectangles c and d are twice as tall as rectangle a and therefore also twice as thin. Hence they have width $\frac{5}{2}$. Rectangle e thus has a width of $\frac{5}{2}+\frac{5}{2}+5=10$ and must be half as tall as rectangle a. This means that rectangle f is precisely $\frac{5}{2}$ times as tall as rectangle a and therefore has a width of $\frac{5}{5/2}=2$. It follows that the square has sides of length $5+\frac{5}{2}+\frac{5}{2}+2=12$. Because the square has a height of $\frac{5}{2}$ times the height of rectangle a, the height of rectangle a equals $|BC|=\frac{12}{5/2}=\frac{24}{5}$.



B2. 56 One sequence consists of pears alone. Next, we count sequences containing at least one apple. In such a sequence, all apples occur consecutively, because there can be no pear anywhere between two apples. If we want to have 8 apples, we can place them in positions 1 through 8, 2 through 9, or 3 through 10. This gives three possible sequences. In this way we find 1 sequence containing 10 apples, 2 sequences containing 9 apples, 3 sequences containing 8 apples, and so on, ending with 10 sequences containing 1 apple. In total there are $1+2+3+\cdots+10=55$ sequences containing at least one apple. The total number of sequences is therefore $55+1=56$.

B3. 18126 A good strategy is to first consider smaller examples. We find

$$\begin{aligned} 9\times 4 &= 40-4 &&= 36,\\ 99\times 44 &= 4400-44 &&= 4356,\\ 999\times 444 &= 444000-444 &&= 443556,\\ 9999\times 4444 &= 44440000-4444 &&= 44435556. \end{aligned}$$

The pattern should be clear. To solve the problem, observe that $999\ldots99=1000\ldots00-1$, where the first number has 2014 zeroes. The product is therefore equal to

$$\underbrace{444\ldots44}_{\text{2014 fours}}\underbrace{000\ldots00}_{\text{2014 zeroes}}-\underbrace{444\ldots44}_{\text{2014 fours}}=\underbrace{444\ldots44}_{\text{2013 fours}}3\underbrace{555\ldots55}_{\text{2013 fives}}6.$$

Adding these digits, we obtain $2013\cdot 4+3+2013\cdot 5+6=2013\cdot 9+9=18126$.

B4. 3 We first show that every pretty table has a score of at least 3. Consider such a table and let a be the number at the very middle. The five numbers in the middle row have an average of a and are not all equal to a. Hence at least one of these numbers must be smaller than a. Similarly, at least one of the numbers in the middle column must be smaller than a. Let this number be b. Since b is the average of the numbers in its row, one of the numbers in that row must be smaller than b, and hence also smaller than a. Thus the table contains at least three different cells that have a number smaller than the number in the very middle. Its score is therefore at least 3.

4	4	3	4	0
4	4	3	4	0
3	3	0	3	−9
4	4	3	4	0
0	0	−9	0	−36

In the figure on the right you can find a pretty table with a score equal to 3. It follows that 3 is the smallest possible score.