51st Dutch Mathematical Olympiad 2012

NEDERLANDSE
WISKUNDE
OLYMPIADE

Contents

Introduction

The selection process for IMO 2013 started with the first round on 27 January 2012, held at the participating schools. The paper consisted of eight multiple choice questions and four open-answer questions, to be solved within 2 hours. In total 5612 students of 270 secondary schools participated in this first round.

Those 811 students from grade 5 (4, $\leq$ 3) that scored 20 (16, 12) points or more on the first round (out of a maximum of 36 points) were invited to the second round, which was held in March at twelve universities in the country. This round contained five open-answer questions, and two problems for which the students had to give extensive solutions and proofs. The contest lasted 2.5 hours.

Those students from grade 5 (4, $\leq$ 3) that scored 32 (27, 22) points or more on the second round (out of a maximum of 40 points) were invited to the final round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited. In total 150 students were invited. They also received an invitation to some training sessions at the universities, in order to prepare them for their participation in the final round.

Out of those 150, in total 143 participated in the final round on 14 September 2012 at Eindhoven University of Technology. This final round contained five problems for which the students had to give extensive solutions and proofs. They were allowed 3 hours for this round. After the prizes had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 51st edition 2012.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad 2012 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June.

Among the participants of the training programme, there were some extra girls, as this year we participated for the second time in the European Girls' Mathematical Olympiad (EGMO). In total there were nine girls competing to be in the EGMO team. The team of four girls was selected by a selection test, held on 8 March 2013. They attended the EGMO in Luxembourg from 8 until 14 April, and the team returned with an honourable mention and a silver medal. For more information about the EGMO (including the 2013 paper), see www.egmo.org.

The same selection test was used to determine the ten students participating in the Benelux Mathematical Olympiad (BxMO), held in Dordrecht, the Netherlands, from 26 until 28 April. The Dutch team managed to come first in the country ranking, and received five bronze medals, two silver medals and two gold medals. For more information about the BxMO (including the 2013 paper), see www.bxmo.org.

In June the team for the International Mathematical Olympiad 2013 was selected by two team selection tests on 5 and 8 June 2013. A seventh, young, promising student was selected to accompany the team to the IMO as an observer C. The team had a training camp in Santa Marta, from 14 July until 21 July.

For younger students the Junior Mathematical Olympiad was held in October 2012 at the VU University Amsterdam. The students invited to participate in this event were the 30 best students of grade 1, grade 2 and grade 3 of the popular Kangaroo math contest. The competition consisted of two one-hour parts, one with fifteen multiple choice questions and one with ten open-answer questions. The goal of this Junior Mathematical Olympiad is to scout talent and to stimulate them to participate in the first round of the Dutch Mathematical Olympiad.

We are grateful to Jinbi Jin and Raymond van Bommel for the composition of this booklet and the translation into English of most of the problems and the solutions.

Dutch delegation

The Dutch team for IMO 2013 in Colombia consists of

- Peter Gerlagh (16 years old)
  - bronze medal at BxMO 2011, honourable mention at BxMO 2012, gold medal at BxMO 2013
  - observer C at IMO 2012
- Ragnar Groot Koerkamp (18 years old)
  - honourable mention at BxMO 2010, gold medal at BxMO 2012, gold medal at BxMO 2013
  - honourable mention at IMO 2011
- Jeroen Huijben (17 years old)
  - bronze medal at BxMO 2010, bronze medal at BxMO 2011
  - observer C at IMO 2010, bronze medal at IMO 2011, gold medal at IMO 2012
- Michelle Sweering (16 years old)
  - bronze medal at EGMO 2012, silver medal at EGMO 2013
  - honourable mention at IMO 2012
- Djurre Tijsma (18 years old)
  - honourable mention at BxMO 2012, silver medal at BxMO 2013
- Jeroen Winkel (16 years old)
  - bronze medal at BxMO 2011, silver medal at BxMO 2012
  - observer C at IMO 2011, bronze medal at IMO 2012

We bring as observer C the promising young student

- Bas Verseveldt (16 years old)
  - silver medal at BxMO 2012, bronze medal at BxMO 2013

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University
- Julian Lyczak (observer B), Utrecht University

Love, fire and mathematics

Meet four passionate fireflies. Anne, Bob, Casey and David.
Their love for each other represents one of the many beautiful mathematical figures that nature can show us. Our fireflies are buzzing around in the spatial shape of a tetrahedron ABCD. Each firefly represents one of the four vertices.
Silent and attractive Anne is head over heels in love with Bob. She is drawn directly towards him. If Bob moves to the right, she moves to the right. If Bob flies up, she flies up. Sporty Bob has a crush too, but not on Anne. If he would have been in love with her, they would fly towards each other and meet in the middle of the edge. That would have been easy! But no, unfortunately Bob is chasing the playful firefly Casey. Casey is a dancer and if she flies to the right, Bob will follow. If she moves down, he moves down. But Casey won't fly towards Bob. She will do everything she can to reach the one she loves: the adventurous David. However, David is crazy about Anne. He flies directly towards her.

All four fireflies are equally in love. Therefore each will fly at the same speed. As time passes the flies will eventually meet in the centre of the tetrahedron. An outburst of passion, lovesickness and heartbreaks follows.

Love is for fools. If they were smart, they all would directly fly to the centre of the figure. But they are not. They each long for their own centre of the world: their loved one.

Question:
What is the distance each firefly will travel before he or she reaches the centre of our geometric form?
Please give the answer in units of the length of an edge of the tetrahedron.

Enjoy and good luck!

If you have found the solution to this puzzle, please send your answer to wiskunde@transtrend.com.

First Round, January 2012

Problems

A-problems

A1. In the multiplication table on the right, the stars represent positive integer numbers such that the table is correct.
What is the largest number that occurs more than once among the entries of the full 5×5-table?

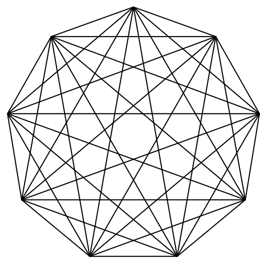
$\times$	$*$	$*$	$*$	7
$*$	24	$*$	$*$	56
$*$	$*$	36	8	$*$
$*$	$*$	27	6	$*$
6	18	$*$	$*$	42

A) 6 B) 8 C) 9 D) 12 E) 18

A2. A palindromic number is a number that does not change when the order of its digits is reversed, like 707 and 154451. Leon lists all 5-digit palindromic numbers (numbers do not start with the digit 0) in increasing order.
Which number is 12th on Leon's list?

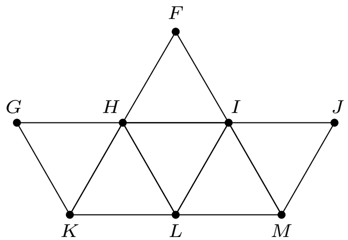
A) 11111 B) 11211 C) 12221 D) 12321 E) 12421

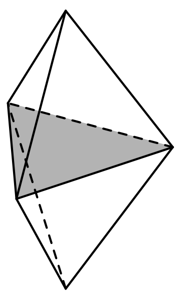
A3. The figure shows a regular nonagon along with all its diagonals. Consider all triangles whose three different vertices are vertices of the nonagon.
How many of these triangles are isosceles?
(An isosceles triangle is a triangle having two or three equal sides.)



A) 27 B) 30 C) 33 D) 36 E) 39

A4.





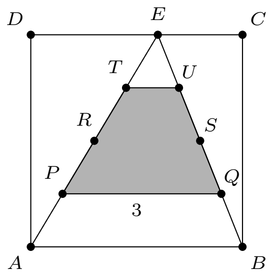
A4. On the previous page is a paper model of a *dipiramid*. The unfolded version is depicted on the left, and the resulting folded dipiramid is depicted on the right.
Which three points of the unfolded model correspond to vertices of the grey triangle?

A) F, G and L B) H, I and M C) G, H and I
D) H, K and L E) F, I and K

A5. Frank has a drawer containing single socks. There are 10 red socks, the other socks are blue. Without looking, he picks a number of socks from the drawer. The number of socks he needs to pick to be sure of getting at least two red socks, is twice the number he needs to pick for getting at least two blue socks.
How many socks does the drawer contain?

A) 14 B) 18 C) 26 D) 32 E) 40

A6. Point E lies on side CD of a square $ABCD$. Line segment AE is divided into four equal parts by points P, R, and T. Line segment BE is divided into four equal parts by points Q, S, and U. The length of PQ equals 3.
What is the area of quadrilateral $PQUT$?



A) $\frac{15}{4}$ B) 4 C) $\frac{17}{4}$ D) $\frac{9}{2}$ E) 5

A7. Carry has six cards. On each card a positive integer number is written. Carry chooses three cards and calculates the sum of the three numbers on these cards. Doing this for each of the 20 possible selections of three cards, she finds the following sums: 16 (ten times) and 18 (also ten times).
What is the smallest number occurring on the cards?

A) 1 B) 2 C) 3 D) 4 E) 5

A8. A sequence of numbers starts as $27, 1, 2012, \dots$ The sequence has the following property: the numbers in positions 1, 2, and 3 add up to 2040. The numbers in positions 2, 3, and 4 add up to 2039, the numbers in positions 3, 4, and 5 add up to 2038, and so on. More generally: the numbers in positions k, $k+1$, and $k+2$ add up to $2041 - k$.
What number is in position 2013?

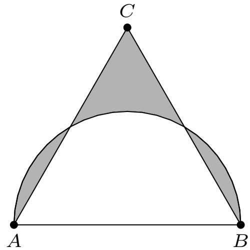
A) -670 B) -669 C) 670 D) 1341 E) 1342

B-problems

The answer to each B-problem is a number.

B1. Consider all 5-digit numbers. Of these numbers, a numbers have the property that the product of their digits is 25, and b numbers have the property that the product of their digits is 15.
Determine $\frac{a}{b}$.

B2. An equilateral triangle ABC has sides of length 12. A semicircle with diameter AB intersects sides AC and BC. The two circular segments outside the triangle, and the part of the triangle outside of the circle are coloured grey.
Determine the total grey area.



B3. A rectangle consists of a number of cells of a regular graphing paper. The number of these cells that touch the boundary is the same as the number of cells that do not touch the boundary.
How many cells does the rectangle contain? Give all possible values.

B4. For all positive numbers x and y, the operation $\star$ obeys the following three rules:

Rule 1: $(2x) \star y = \frac{1}{2} + (x \star y)$.

Rule 2: $y^2 \star x = x^2 \star y$.

Rule 3: $2 \star 2 = \frac{3}{2}$.

Compute $32 \star 8$.

Solutions

A-problems

A1. D) 12 Consider the top figure. The cell marked by the star must contain a number that divides both 27 and 6. Hence this number is either 1 or 3. The first option implies that the cell marked by the double star contains the number 27. But 36 is not divisible by 27, so this option is off.
Using the second option, we can complete the full table and arrive at the bottom figure. Clearly, 12 is the largest number occurring more than once.

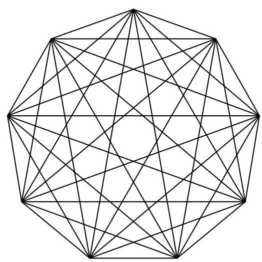
×		**		7
	24			56
		36	8	
*		27	6	
6	18			42

×	3	9	2	7
8	24	72	16	56
4	12	36	8	28
3	9	27	6	21
6	18	54	12	42

A2. A) 11111 A 5-digit palindromic number is precisely determined by specifying the first three digits: the last digit then equals the first digit, and the second last digit equals the second digit. Therefore, the first twelve 5-digit palindromic numbers are:

$$\mathbf{100}01, \mathbf{101}01, \mathbf{102}01, \ldots, \mathbf{109}01, \mathbf{110}11, \mathbf{111}11.$$

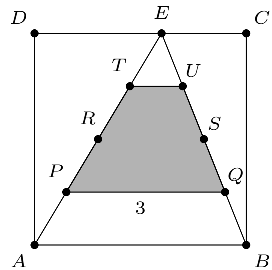
A3. B) 30 We start by counting the number of equilateral triangles: there are 3 of those. Next, we count the number of isosceles triangles that are not equilateral. Such a triangle has 9 possibilities for its apex. For each choice of the apex, there are 3 possibilities for its base. This gives a total of $9 \times 3 = 27$ possibilities. Hence, in total we have $3 + 27 = 30$ isosceles triangles.



A4. B) H, I and M Different letters may represent the same vertex of the dipiramid. The five vertices are $G = F = J$, $K = M$, H, I and L. Since H and I both have four neighbours, they represent vertices of the grey triangle. Also $K = M$ has four neighbours: I, H, L and $G = F = J$, so that must be the third vertex of the grey triangle.

A5. D) 32 Denote the number of blue socks in the drawer by b. To be sure of getting two *blue* socks, Frank must take at least 12 socks. Indeed, in the worst case he will start by picking the ten red socks. To be sure of getting at least two *red* socks, he must take at least $b+2$ socks, because in the worst case he will first pick all b blue socks.
Since we know that this second number is twice the first number, we find that $b+2 = 2 \times 12 = 24$. This implies that $b = 22$. The total number of socks is therefore $22 + 10 = 32$.

A6. B) 4 Triangles EAB, EPQ, and ETU are similar since $|AE| : |PE| : |TE| = 4 : 3 : 1 = |BE| : |QE| : |UE|$ and $\angle AEB = \angle PEQ = \angle TEU$ (SAS). This implies that $|AB| : |PQ| : |TU| = 4 : 3 : 1$. It follows that $|AB| = 4$ and hence the area of triangle ABE equals $\frac{1}{2} \times 4 \times 4 = 8$.
Triangles PQE and TUE must therefore have area $(\frac{3}{4})^2 \times 8 = \frac{9}{2}$ and $(\frac{1}{4})^2 \times 8 = \frac{1}{2}$, respectively, by similarity. It follows that the area of the quadrilateral $PQUT$ is equal to $\frac{9}{2} - \frac{1}{2} = 4$.



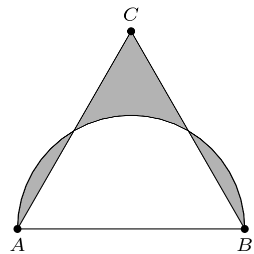
A7. D) 4 Because only two different outcomes occur, there can be no more than two different numbers on the cards. Indeed, if there would be three cards with different numbers, they would give three different outcomes when combined with a pair of the remaining three cards.
Let us denote the two numbers on the cards by a and b. Without loss of generality, we can assume that a occurs on at least three cards. Because $a+a+a$, $a+a+b$, and $a+b+b$ are different, the number b can occur only once.
There are two cases: the case $a+a+a = 16, a+a+b = 18$ and the case $a+a+a = 18, a+a+b = 16$. The first case is off because 16 is not a multiple of 3 (a is an integer). Hence $a = \frac{18}{3} = 6$ and $b = 16 - 12 = 4$. We conclude that 4 is the smallest number occurring on the cards.

A8. E) 1342 For any four consecutive terms a, b, c, d in the sequence, we have $d = a - 1$. This follows from the fact that $b + c + d = (a + b + c) - 1$ by the given property of the sequence. In other words: skipping forward in the sequence by three positions, decreases the current number by one. Hence in positions $3, 6, 9, \ldots, 2013$ $(= 3 + 3 \times 670)$ we find the numbers $2012, 2011, \ldots, (2012 - 670 =)\ 1342$.

B-problems

B1. $\frac{1}{2}$ The 5-digit numbers with product of the digits equal to 25, consist of two fives and three ones. There are a of these numbers, where a is the number of ways to place the three ones (because the positions of the fives is then fixed as well).
The 5-digit numbers with product of the digits equal to 15, consist of one three, one five, and three ones. There are b of those numbers, where b equals the number of ways to first place the three ones, and then place the digits 3 and 5 in one of two possible ways.
We see that $b = 2a$, which implies that $\frac{a}{b} = \frac{1}{2}$.

B2. 6π Point F is the midpoint of AB (and the center of the semicircle). Points D and E are the midpoints of BC and AC. They lie on the semicircle because triangles BDF and AEF are equilateral. Draw the arc between D and E of the circle with center C and radius $|CE| = 6$. Because $\angle AFE = \angle EFD = 60°$, the circular segments on top of AE and on top of DE are congruent. The circular segment below DE is congruent to the other two, because the circles around C and F have the same radius. It follows that the three grey regions together have the same area as the circle sector CED, which has an area of $\frac{1}{6}(\pi \cdot 6^2) = 6\pi$.



B3. 48 and 60 Let a and b denote the number of cells in the length and width of the rectangle. We may assume that $a \geq b$. Therefore, the total number of cells in the rectangle is ab and the number of cells at the edges equals $2a + 2b - 4$. Given the fact that half of the cells are at the edge of the rectangle, we know that $ab = 2(2a + 2b - 4)$. Rearranging terms gives $ab - 4a - 4b + 16 = 8$. Factoring the left-hand side, we get $(a-4)(b-4) = 8$. Since a and b are positive integers and $a \geq b$, the only possibilities are $a - 4 = 8, b - 4 = 1$ and $a - 4 = 4, b - 4 = 2$ (note that $a - 4$ and $b - 4$ cannot be negative because in that case $b - 4$ is at most -4, contradicting the fact that b is positive). We find $a = 12, b = 5$ or $a = 8, b = 6$. This gives two possible rectangles with $12 \times 5 = 60$ and $8 \times 6 = 48$ cells respectively.

B4. $\frac{11}{2}$ Using **Rule 1** we find: $32 \triangle 8 = \frac{1}{2} + 16 \triangle 8 = \frac{2}{2} + 8 \triangle 8 = \frac{3}{2} + 4 \triangle 8$.
From **Rule 2** with $y = 2$ and $x = 8$ it follows that : $4 \triangle 8 = 64 \triangle 2$.
Repeated application of **Rule 1** gives: $64 \triangle 2 = \frac{1}{2} + 32 \triangle 2 = \frac{2}{2} + 16 \triangle 2 = \ldots = \frac{5}{2} + 2 \triangle 2$.
Collecting these results and using **Rule 3**, we obtain:

$$32 \triangle 8 = \frac{3}{2} + 4 \triangle 8 = \frac{3}{2} + 64 \triangle 2 = \frac{8}{2} + 2 \triangle 2 = \frac{11}{2}.$$

Second Round, March 2012

Problems

B-problems

The answer to each B-problem is a number.

B1. In this addition, each letter represents a digit (0 to 9). Different letters represent different digits. Determine the value of W $\times$ R.

$$
\begin{array}{cccccc}
\mathrm{T} & \mathrm{W} & \mathrm{E} & \mathrm{E} & \mathrm{D} & \mathrm{E} \\
 & \mathrm{R} & \mathrm{O} & \mathrm{N} & \mathrm{D} & \mathrm{E}\ + \\
\hline
2 & 3 & 0 & 3 & 1 & 2
\end{array}
$$

B2. All 2012 camels in the Netherlands are to be distributed among 40 pastures. No two pastures are allowed to get the same number of camels. The pasture in the city centre of Amsterdam has to get the largest number of camels. At least how many camels have to be placed in that pasture?

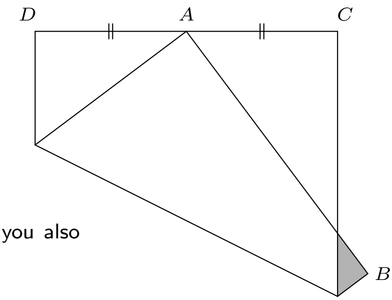
B3. One of the four dwarfs Anne, Bert, Chris and Dirk stole the king's gold. The dwarfs, who know each other very well, each make two statements. If a dwarf is a liar, then at least one of these two statements is a lie. If the dwarf is not a liar, then both statements are true.

Anne says: "Bert is a liar." and "Chris or Dirk stole the gold."
Bert says: "Chris is a liar." and "Dirk or Anne stole the gold."
Chris says: "Dirk is a liar." and "Anne or Bert stole the gold."
Dirk says: "Anne is a liar." and "Bert or Chris stole the gold."

How many of these eight statements are true?

B4. On each of the 10,000 squares of a 100$\times$100-chess board a number is written. Along the top row the numbers 0 to 99 are written from left to right. In the left column the numbers 0 to 99 are written from top to bottom. The sum of the four numbers in a 2$\times$2-block always equals 20.
What number is written in the bottom right square of the board?

B5. A square $ABCD$ with side length 8 is folded in such a way that vertex A becomes the midpoint of CD (see figure).
Find the area of the grey triangle.



C-problems

For the C-problems not only the answer is important; you also have to describe the way you solved the problem.

C1. You have one card with the number 12 on it. You are allowed to add new cards to your collection according to the following rules.

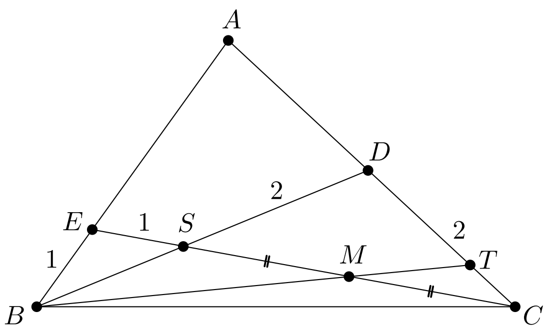
- If you have a card with the number a on it, then you are allowed to make a new card with the number $2a + 1$ on it.
- If you have a card with the number b on it and b is divisible by 3, then you are allowed to make a new card with the number $\frac{b}{3}$ on it.

(a) Show that you can make a card with the number 29 on it.

(b) Show that you can make make a card with the number $2^{2012} - 1$ on it.

(c) Show that you can never make a card with the number 100 on it.

C2. Given is a triangle ABC, a point D on line segment AC and a point E on line segment AB. The intersection of BD and CE is called S. The midpoint of line segment CS is called M. The line BM intersects line segment CD in point T. Finally we are given that $|BE| = |ES| = 1$ and $|CD| = |DS| = 2$.
Prove that $|AB| = |AT|$.



Solutions

B-problems

B1. 32 We will reason through the addition from right to left. Either $2E = 12$ or $2E = 2$. The second case is excluded because then $2D$ would equal 1 or 11. Therefore $E = 6$.

$$\begin{array}{cccccc} T & W & E & E & D & E \\ & R & O & N & D & E + \\ \hline 2 & 3 & 0 & 3 & 1 & 2 \end{array}$$

Either $D = 0$ or $D = 5$. The second case is excluded because then $E + N + 1 = 13$ or $N = 6$ would hold, but 6 is already taken by E. So $D = 0$ and $E + N = 13$, which implies that $N = 7$.

From $E + O + 1 = 10$ it follows that $O = 3$. Hence $W + R = 12$ or $W + R = 2$. The second case is excluded: $W + R \geq 1 + 2 = 3$ because 0 is already taken. Hence, $W + R = 12$ (and $T = 1$).

The pair $\{W, R\}$ must be one of the following: $\{3, 9\}$, $\{4, 8\}$, and $\{5, 7\}$. The first and last possibility are excluded because 3 and 7 are already taken. We conclude that $W \cdot R = 8 \cdot 4 = 32$.

B2. 70 We may put 23 camels in one pasture, and put $32, 33, 34, \ldots,$ 70 camels in the remaining 39 pastures, respectively. The last pasture is the one in the centre of Amsterdam. The total number of assigned camels indeed equals $23+(32+33+\cdots+70) = 23+39\cdot\frac{32+70}{2} = 2012$. Hence, a valid distribution having 70 camels in the pasture in the centre of Amsterdam exists.

It is not possible to do it with less than 70. Indeed, suppose that at most 69 camels are put in the pasture in the centre of Amsterdam. Because the number of camels is different for each pasture, the second most populated pasture has at most 68 camels, the third most populated pasture has at most 67 camels, and so on. In total the pastures have at most $69 + 68 + \ldots + 30 = 40 \cdot \frac{30+69}{2} = 1980$ camels, leaving at least 32 unassigned camels. Therefore, no solution exists with fewer than 70 camels in the pasture in the centre of Amsterdam. The minimum is therefore 70.

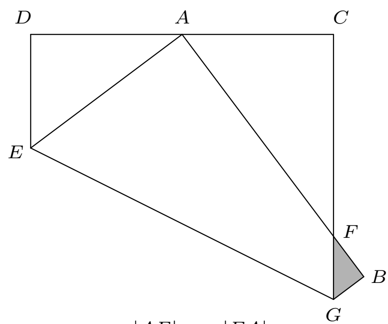
B3. 5 First consider the case that Anne has stolen the king's gold. The last statement of both Bert and Chris is true in this case, and the last statement of both Anne and Dirk is false. Hence, Anne and Dirk are liars. Since both of Chris' statements are true, Chris is no liar. This implies that Bert is a liar, because his first statement was a lie. Now that we know who is a liar and who is not, it is easy to see that exactly five of the eight

statements are true.
In the cases where Bert, Chris or Dirk is the thief, a similar reasoning holds. By the symmetry of the problem (cyclically permuting the names 'Anne', 'Bert', 'Chris', and 'Dirk' does not change the problem), exactly five of the statements are true in each case.

B4. -178 Colour the 10,000 squares in a chess board pattern: the upper left square and bottom right square will be white and the squares in the upper right and bottom left will be black. Consider the $99 \cdot 99 = 9801$ 2×2-blocks. The 4901 blocks having a white square in the upper left corner are called the *white blocks* and the 4900 blocks having a black square in the upper left corner are called the *black blocks*. For each white block, add the four numbers it contains. Let W be the result of adding these 4901 outcomes (some squares are counted twice!). Let Z be the result when applying the same procedure to the 4900 black blocks. Since the number of white blocks is one more than the number of black blocks, we have $W - Z = 20$.

Now consider for each square how many times it is counted in total. Each of the four corner squares is counted only in one white block. Each of the other squares at the edge of the board are counted in exactly one white block and one black block. Each of the remaining squares is counted in exactly two white blocks and two black blocks. Considering the difference $W - Z$, only the four corner squares are counted, each exactly once. For the number x in the bottom right corner, we find that $x + 0 + 99 + 99 = W - Z = 20$, hence $x = -178$.

B5. $\frac{2}{3}$ Introduce points E, F, and G as in the figure. Suppose that $|DE| = x$. Then $|AE| = 8-x$, because the square has sides of length 8. Using the Pythagorean theorem, we get $(8-x)^2 = |AE|^2 = |DE|^2 + |AD|^2 = x^2 + 16$. Solving this quadratic equation gives $x = 3$. Observe that we have $\angle CAF = 180° - \angle DAE - \angle EAB = 90° - \angle DAE = \angle DEA$. Also, $\angle ADE = 90° = \angle FCA$. Therefore, triangles DEA and CAF are similar (AA). This implies that $\frac{1}{4}|AF| = \frac{|AF|}{|AC|} = \frac{|EA|}{|DE|} = \frac{5}{3}$, and hence $|AF| = \frac{20}{3}$. We also see that $\frac{1}{4}|CF| = \frac{|CF|}{|AC|} = \frac{|AD|}{|DE|} = \frac{4}{3}$ and so $|CF| = \frac{16}{3}$. It follows that $|BF| = 8 - |AF| = \frac{4}{3}$. Since $\angle CFA = \angle BFG$ (vertical angles), and $\angle ACF = \angle GBF = 90°$, triangles CFA and BFG



are similar (AA). This implies that $\frac{3}{4}|BG| = \frac{|BG|}{|BF|} = \frac{|AC|}{|CF|} = \frac{4}{\frac{16}{3}} = \frac{3}{4}$ and therefore $|BG| = 1$. It follows that the area of the grey triangle equals $\frac{1}{2} \cdot |BG| \cdot |BF| = \frac{1}{2} \cdot 1 \cdot \frac{4}{3} = \frac{2}{3}$.

C-problems

C1. (a) Applying the two rules, we can make the following sequence of cards: $12 \to 4 \to 9 \to 3 \to 7 \to 15 \to 31 \to 63 \to 21 \to 43 \to 87 \to 29$.

(b) We have seen in part (a) how to make a card with the number $3 = 2^2 - 1$. Iterating the first rule, we can sequentially construct cards with numbers $2 \cdot (2^2 - 1) + 1 = 2^3 - 1$, $2 \cdot (2^3 - 1) + 1 = 2^4 - 1$, and so on. In particular, we can make the number $2^{2012} - 1$.

(c) Applying rule 1 to any card produces a card with an *odd* number. Applying rule 2 to a card with an *odd* number (assuming it is a multiple of three), produces a card with an *odd* number. As soon as we apply rule one, the resulting card will only give rise to cards with an *odd* number. To get *even* numbers, we must therefore restrict to using rule 2 only (repeatedly). The only *even* numbers that can be made are therefore 12 and 4.

C2. Observe that $\angle ESB = \angle DSC$ (vertical angles). Since triangles BES and SDC are isosceles, $\angle EBS = \angle ESB = \angle DSC = \angle DCS$. Hence triangles BES and SDC are similar (AA). In particular, $|BS| = \frac{|BS|}{|BE|} = \frac{|SC|}{|SD|} = \frac{1}{2}|SC| = |SM|$. Therefore, triangle BSM is isosceles and $\angle SBM = \angle SMB = \angle TMC$ (vertical angles). Using that the angles of a triangle sum to $180°$, we find that $\angle TMC = 180° - \angle MTC - \angle TCM$ and hence also $\angle ATB = 180° - \angle MTC = \angle TMC + \angle TCM$. We had already shown that $\angle TMC = \angle SBM$ and $\angle TCM = \angle DCS = \angle ABS$. We therefore see that $\angle ATB = \angle SBM + \angle ABS = \angle ABT$. It follows that triangle BAT is isosceles with top A, and therefore $|AB| = |AT|$ holds.

Final Round, September 2012

Problems

For these problems not only the answer is important; you also have to describe the way you solved the problem.

1. Let a, b, c, and d be four distinct integers.
Prove that $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.

2. We number the columns of an $n \times n$-board from 1 to n. In each cell, we place a number. This is done in such a way that each row precisely contains the numbers 1 to n (in some order), and also each column contains the numbers 1 to n (in some order). Next, each cell that contains a number greater than the cell's column number, is coloured grey. In the figure below you can see an example for the case $n = 3$.

1	2	3
3	1	2
1	2	3
2	3	1

(a) Suppose that $n = 5$. Can the numbers be placed in such a way that each row contains the same number of grey cells?

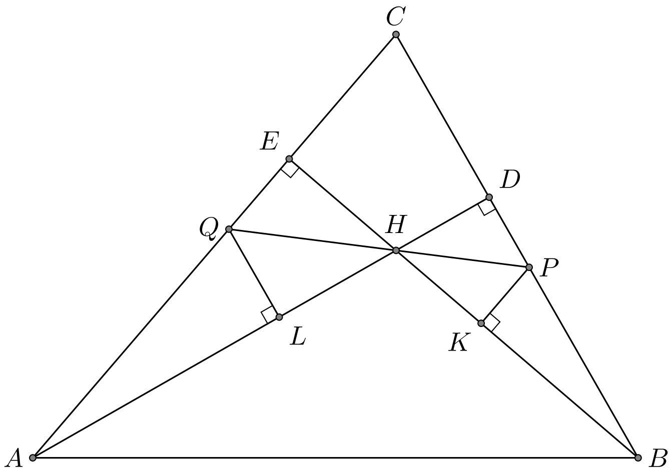
(b) Suppose that $n = 10$. Can the numbers be placed in such a way that each row contains the same number of grey cells?

3. Determine all pairs (p, m) consisting of a prime number p and a positive integer m, for which

$$p^3 + m(p+2) = m^2 + p + 1$$

holds.

4. We are given an acute triangle ABC and points D on BC and E on AC such that AD is perpendicular to BC and BE is perpendicular to AC. The intersection of AD and BE is called H. A line through H intersects line segment BC in P, and intersects line segment AC in Q. Furthermore, K is a point on BE such that PK is perpendicular to BE, and L is a point on AD such that QL is perpendicular to AD.



Prove that DK and EL are parallel.

5. The numbers 1 to 12 are arranged in a sequence. The number of ways this can be done equals $12 \times 11 \times 10 \times \cdots \times 1$. We impose the condition that in the sequence there should be exactly one number that is smaller than the number directly preceding it.
How many of the $12 \times 11 \times 10 \times \cdots \times 1$ sequences satisfy this condition?

Solutions

1. In order to show that the product

$$n = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12, it suffices to show that it is divisible by 3 and by 4. We consider the remainder upon division by 3 for the numbers a, b, c, and d. Two of these numbers must have the same remainder since there are four numbers and only three possible remainders. The difference of these two numbers must then be divisible by 3, which implies that also n is divisible by 3.

If among the numbers a, b, c, and d there are at least three numbers of the same parity (all three *even* or all three *odd*), this implies that the three pairwise differences between these numbers are even. Hence n is divisible by $2\times 2\times 2 = 8$.

Otherwise, two of the numbers are *even* and the other two are *odd*. Both pairwise differences are *even*, which shows that n is divisible by 4. □

2. (a) Yes, it can be done. A possible filling of the 5×5-board containing exactly two grey cells in each row can be seen in the figure below.

1	2	3	4	5
5	4	3	2	1
1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5

(b) Consider a 10×10-board that is filled in accordance to the given rules. Every column now contains the numbers 1 to 10. In column 1, there are 9 grey cells (those containing numbers ‘2’ to ‘10’), in column 2 there are 8 grey cells (containing ‘3’ to ‘10’), in column 3 there are 7 grey cells (containing ‘4’ to ‘10’), etc. In total, there are $9+8+7+\cdots+0 = 45$ grey cells. Because 45 is not divisible by 10, it is impossible for all rows to have the same number of grey cells.

□

3. All terms containing a factor p are brought to the left-hand side of the equation. In this way we obtain $p^3 + mp - p = m^2 - 2m + 1$, or

$$p(p^2 + m - 1) = (m - 1)^2.$$

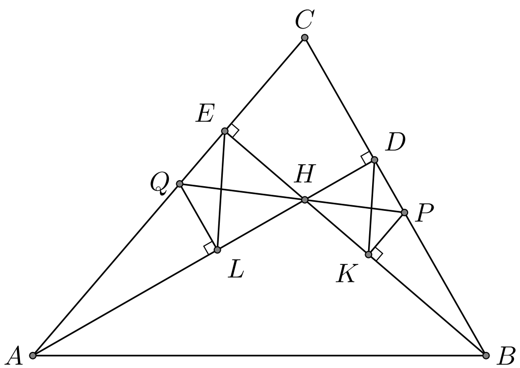
Note that $m-1$ is non-negative by assumption. Observe that p is a divisor of $(m-1)^2$. Since p is a prime number, p must also divide $m-1$. We may write $m-1 = kp$, with k a non-negative integer. Substituting this into the previous equation gives: $p(p^2 + kp) = k^2p^2$. Dividing by p^2 on both sides, we find $p + k = k^2$, or

$$p = k(k - 1).$$

As p is prime, one of the factors k and $k-1$ must equal 1 (the case $k - 1 = -1$ is excluded). The case $k = 1$ does not lead to a solution because then $k - 1 = 0$. Hence, we must have $k - 1 = 1$, which gives the only candidate for a solution $k = 2$, $p = 2$ en $m = 5$, and hence $(p, m) = (2, 5)$. It is clear that this is indeed a solution. $\square$

4. We will use similarity of certain triangles. Observe that $\angle DHP = \angle LHQ$ (opposite angles), and that $\angle PDH = 90° = \angle QLH$. It follows that triangles DHP and LHQ are similar (AA). This implies that $\frac{|DH|}{|LH|} = \frac{|HP|}{|HQ|}$. In the same way, we can see that triangles KHP and EHQ are similar, which implies that $\frac{|KH|}{|EH|} = \frac{|HP|}{|HQ|}$.

Combining these two inequalities shows that $\frac{|DH|}{|LH|} = \frac{|KH|}{|EH|}$. Because we have $\angle DHK = \angle LHE$ (opposite angles), this implies that triangles DHK and LHE are similar (SAS). Similarity of these triangles implies that $\angle HKD = \angle HEL$, from which we may conclude (corresponding angles) that DK and EL are parallel. $\square$



5. A sequence satisfying the condition is called 'good'. We will determine the number of good sequences in an indirect way, by first counting something else.

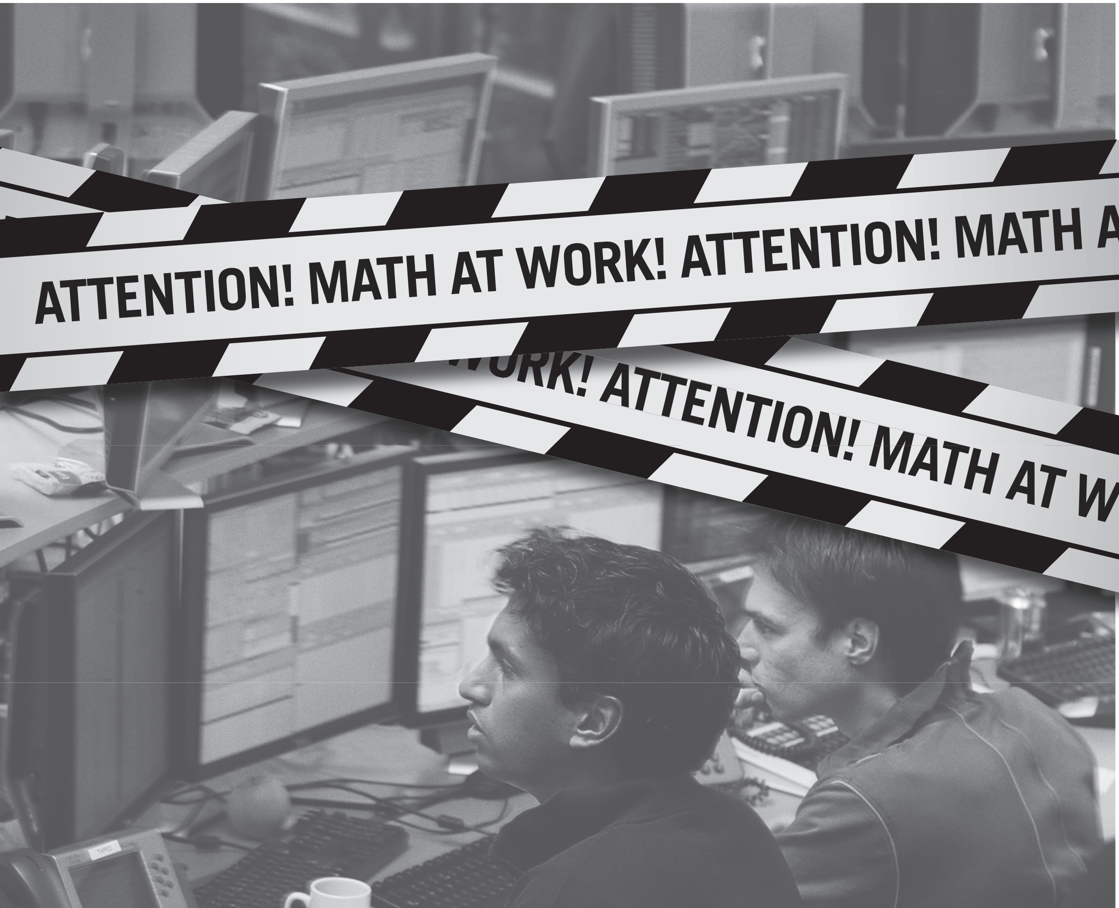
First consider the number of ways in which we can colour the numbers 1 to 12, each number being coloured either red or blue. As there are two options (red or blue) for each of the 12 numbers independently, there are 2^{12} possible colourings. We call a colouring 'good' if there is at least one number of each colour, and the largest red number is larger than the smallest blue number. There are exactly 13 colourings that are *not* good: the colouring having only blue numbers, and the twelve colourings in which the red numbers are precisely the numbers 1 to k, for some $k = 1, 2, \ldots, 12$ (in the case $k = 12$, there are no blue numbers.) In total, there are $2^{12} - 13 = 4083$ good colourings.

We will now show that the number of good sequences is equal to the number of good colourings, by perfectly matching colourings and sequences. Take a good sequence and suppose that a is the number smaller than its immediate predecessor in the sequence. To this sequence we associate the following colouring: the numbers preceding a in the sequence are coloured red, and the other numbers are coloured blue. This will be a good colouring.

As an example, consider the good sequence $1, 3, 4, 5, 8, 9, 2, 6, 7, 10, 11, 12$. Hence, $a = 2$. The numbers 1, 3, 4, 5, 8, and 9 will be coloured red, the numbers 2, 6, 7, 10, 11, and 12 will be blue. This is a good colouring, because $9 > 2$.

In this way, each good colouring is obtained precisely once from a good sequence. Indeed, for a given good colouring, we can find the unique corresponding good sequence as follows: write down the red numbers in increasing order, followed by the blue numbers in increasing order.

We conclude that there are 4083 sequences satisfying the condition. $\square$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BxMO/EGMO Team Selection Test, March 2013

Problems

1. In quadrilateral $ABCD$ the sides AB and CD are parallel. Let M be the midpoint of diagonal AC. Suppose that triangles ABM and ACD have equal area. Prove that $DM \parallel BC$.

2. Consider a triple (a, b, c) of pairwise distinct positive integers satisfying $a + b + c = 2013$. A *step* consists of replacing the triple (x, y, z) by the triple $(y + z - x, z + x - y, x + y - z)$. Prove that, starting from the given triple (a, b, c), after 10 steps we obtain a triple containing at least one negative number.

3. Find all triples (x, n, p) of positive integers x and n and primes p for which the following holds:

$$x^3 + 3x + 14 = 2 \cdot p^n.$$

4. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f\big(x + yf(x)\big) = f\big(xf(y)\big) - x + f\big(y + f(x)\big)$$

for all $x, y \in \mathbb{R}$.

5. Let $ABCD$ be a cyclic quadrilateral for which $|AD| = |BD|$. Let M be the intersection of AC and BD. Let I be the incentre (centre of the inscribed circle) of $\triangle BCM$. Let N be the second point of intersection of AC and the circumscribed circle of $\triangle BMI$. Prove that $|AN| \cdot |NC| = |CD| \cdot |BN|$.

Solutions

1. Because M is the midpoint of AC, the area of triangle ABM is equal to the area of triangle BCM. Hence the area of triangle ABC is two times the area of triangle ABM and hence it is also two times the area of triangle ACD. The height of triangle ACD with respect to the basis CD is the distance between the parallel lines AB and CD. That is also the height of triangle ABC with respect to the basis AB. Because these heights are equal, we have $|AB| = 2 \cdot |CD|$.

Let K be the midpoint of AB. Then KM is the line segment connecting the midpoints of AB and AC and hence $KM \parallel BC$. Furthermore we have $|KB| = \frac{1}{2}|AB| = |CD|$, hence quadrilateral $KBCD$ has a pair of parallel sides and a pair of sides that have the same length, which proves that it is a parallelogram. Hence $DK \parallel BC$. But this means that DK and KM are the same line (because they both pass through K and are parallel to BC) and this line is the line DM. Hence, DM is parallel to BC. □

2. Because the sum of a, b and c is equal to 2013, we can write

$$(a, b, c) = (671 + u, 671 + v, 671 + w)$$

with $u + v + w = 0$. Let (a_i, b_i, c_i) be the triple that one obtains after applying i steps. After one step, the first number is replaced by

$$a_1 = b + c - a = 671 - 2u.$$

The difference with 671 is multiplied by -2 after one step. The same holds for the second and third number of the triple. By using induction we get

$$(a_i, b_i, c_i) = (671 + (-2)^i \cdot u, 671 + (-2)^i \cdot v, 671 + (-2)^i \cdot w).$$

Because $u + v + w = 0$ and u, v and w must be pairwise distinct, one of them must be negative, say that u is negative. Then

$$a_{10} = 671 + (-2)^{10} \cdot u \leq 671 - 1024 < 0.$$

□

3. The left hand side can be factorised as $(x + 2)(x^2 - 2x + 7)$. The equation becomes

$$(x + 2)(x^2 - 2x + 7) = 2p^n.$$

First consider the case in which x is even. Then $x+2 = 2p^a$ for a certain integer $a \geq 0$ and $x^2 - 2x + 7 = p^{n-a}$. (Notice that this also holds in the case in which $p = 2$.) Because x is a positive integer, we have $x+2 \geq 3$ and we can rule out $a = 0$. Furthermore we have $x^2 - 3x + 5 > (x - \frac{3}{2})^2 \geq 0$ for all x, hence $x^2 - 2x + 7 > x + 2$. This yields that $n - a > a$. We now substitute $x = 2p^a - 2$ in $x^2 - 2x + 7 = p^{n-a}$:

$$4p^{2a} - 12p^a + 15 = (2p^a - 2)^2 - 2(2p^a - 2) + 7 = p^{n-a}.$$

Because $n - a > a$, we have $p^a \mid p^{n-a}$. Furthermore $p^a \mid p^a$ and $p^a \mid p^{2a}$ hold. Hence p^a is a divisor of 15. This yields $p = 3$ or $p = 5$ and furthermore $a = 1$. If $p = 3$, then the left hand side is 15 and the right hand side is 3^{n-1}; that equation has no solution. If $p = 5$, the left hand side is 55 and the right hand side is 5^{n-1}; this equation also has no solution.

Now consider the case in which x is odd. Then $x + 2 = p^a$ holds for a certain integer $a \geq 0$ and $x^2 - 2x + 7 = 2p^{n-a}$. Because x is a positive integer, we have $x + 2 \geq 3$ and we can rule out $a = 0$ again. Furthermore $x \neq 2$ holds, hence also $x^2 - 4x + 3 = (x-2)^2 - 1 \geq 0$, hence $x^2 - 2x + 7 \geq 2(x + 2)$. Therefore we have $n - a \geq a$. We substitute $x = p^a - 2$ in $x^2 - 2x + 7 = 2p^{n-a}$:

$$p^{2a} - 6p^a + 15 = 2p^{n-a} = (p^a - 2)^2 - 2(p^a - 2) + 7 = 2p^{n-a}.$$

Because $n - a \geq a$, we have $p^a \mid p^{n-a}$. Hence, also in this case we find $p^a \mid 15$, which yields $p = 3$ or $p = 5$ and furthermore $a = 1$. If $p = 3$, the left hand side is 6 and the right hand side is $2 \cdot 3^{n-1}$, hence $n = 2$. We find $x = 3 - 2 = 1$. Indeed, $(1, 2, 3)$ is a solution. If $p = 5$, the left hand side is 10 and the right hand side is $2 \cdot 5^{n-1}$, hence $n = 2$. We find $x = 5 - 2 = 3$. Indeed, $(3, 2, 5)$ is a solution.

There are two solutions, namely $(1, 2, 3)$ and $(3, 2, 5)$. □

4. Substituting $x = y = 0$ gives

$$f(0) = f(0) - 0 + f\big(f(0)\big),$$

hence $f\big(f(0)\big) = 0$. Substituting $x = y = 1$ gives

$$f\big(1 + f(1)\big) = f\big(f(1)\big) - 1 + f\big(1 + f(1)\big),$$

hence $f\big(f(1)\big) = 1$. Now substitute $x = 1$ and $y = 0$ to find

$$f(1) = f\big(f(0)\big) - 1 + f\big(f(1)\big).$$

We know that $f\big(f(0)\big) = 0$ and $f\big(f(1)\big) = 1$, hence we now find $f(1) = 0$. Because $f\big(f(1)\big) = 1$, this also yields that $f(0) = 1$. Now substitute $y = 0$ to get

$$f(x) = f(x) - x + f\big(f(x)\big) \qquad \text{for all } x \in \mathbb{R},$$

hence $f\big(f(x)\big) = x$ for all $x \in \mathbb{R}$. Substitute $x = 1$ to get

$$f(1) = f\big(f(y)\big) - 1 + f(y) \qquad \text{for all } x \in \mathbb{R}.$$

Together with $f(1) = 0$ and $f\big(f(y)\big) = y$ we now find $0 = y - 1 + f(y)$, hence $f(y) = 1 - y$ for all $y \in \mathbb{R}$. Now we know for sure that $f(x) = 1 - x$ for all $x \in \mathbb{R}$ is the only possible solution to the equation. If we substitute this in the original equation both the left and right hand side are equal to

$$1 - x - y + xy.$$

Hence this function is a solution and it is the only solution. $\square$

5. Let $\alpha = \angle DAB$. Because $|AD| = |BD|$, we also have $\angle ABD = \alpha$. By the inscribed angle theorem we find $\angle ACD = \alpha$, while the cyclic quadrilateral theorem yields that $\angle BCD = 180° - \alpha$. Hence $\angle BCA = 180° - 2\alpha$. The angle sum theorem in triangle BIM yields, together with the fact that I is the intersection point of the angular bisectors of $\triangle BCM$, that

$$\begin{aligned}\angle BIM = 180° - \angle IMB - \angle MBI &= 90° + 90° - \tfrac{1}{2}\angle CMB - \tfrac{1}{2}\angle MBC \\ = 90° + \tfrac{1}{2}\angle BCM = 90° + \tfrac{1}{2}\angle BCA &= 90° + 90° - \alpha = 180° - \alpha.\end{aligned}$$

Because $BIMN$ is a cyclic quadrilateral, this yields $\angle BNM = \alpha$. The angle sum theorem in $\triangle BNC$ now yields

$$\begin{aligned}\angle NBC = 180° - \angle BCN - \angle CNB &= 180° - \angle BCA - \angle MNB \\ &= 180° - (180° - 2\alpha) - \alpha = \alpha.\end{aligned}$$

This means that

$$\angle ABN = \angle ABC - \angle NBC = \angle ABC - \alpha = \angle ABC - \angle ABD = \angle CBD.$$

This fact combined with $\angle NAB = \angle CAB = \angle CDB$, which holds by the inscribed angle theorem, yields $\triangle ABN \sim \triangle DBC$ (AA). Hence

$$\frac{|AN|}{|CD|} = \frac{|BN|}{|CB|},$$

or equivalently $|CD| \cdot |BN| = |AN| \cdot |CB|$. We know that $\angle NBC = \alpha = \angle BNM = \angle BNC$, hence $\triangle BNC$ is isosceles with apex C, hence $|CB| = |CN|$. Hence we have $|CD| \cdot |BN| = |AN| \cdot |CN|$ and this is what we wanted to prove. $\square$

IMO Team Selection Test 1, June 2013

Problems

1. Determine all 4-tuples (a, b, c, d) of real numbers satisfying the following four equations.

$$\begin{aligned} ab + c + d &= 3, \\ bc + d + a &= 5, \\ cd + a + b &= 2, \\ da + b + c &= 6. \end{aligned}$$

2. Determine all integers n for which $\frac{4n-2}{n+5}$ is the square of a rational number.

3. Fix a triangle $\triangle ABC$. Let Γ_1 be the circle through B tangent to edge AC in A. Let Γ_2 be the circle through C tangent to edge AB in A. The second intersection of Γ_1 and Γ_2 is denoted by D. The line AD has second intersection E with the circumcircle of $\triangle ABC$. Show that D is the midpoint of the segment AE.

4. Let $n \geq 3$ be an integer, and consider a $n \times n$-board, divided into n^2 unit squares. For all $m \geq 1$, arbitrarily many $1 \times m$-rectangles (type I) and arbitrarily many $m \times 1$-rectangles (type II) are available. We cover the board with N such rectangles, without overlaps, and such that every rectangle lies entirely inside the board. We require that the number of type I rectangles used is equal to the number of type II rectangles used. (Note that a 1×1-rectangle has *both* types.) What is the minimal value of N for which this is possible?

5. Let a, b, and c be positive real numbers satisfying $abc = 1$. Show that

$$a + b + c \geq \sqrt{\tfrac{1}{3}(a+2)(b+2)(c+2)}.$$

Solutions

1. Subtracting the first equation from the second one yields

$$2 = 5 - 3 = (bc + d + a) - (ab + c + d) = b(c - a) + a - c = (b - 1)(c - a).$$

Subtracting the third equation from the fourth one yields

$$4 = 6 - 2 = (da + b + c) - (cd + a + b) = d(a - c) + c - a = (1 - d)(c - a).$$

We deduce that $c - a \neq 0$, and hence that $1 - d = 2(b - 1)$, or equivalently, $3 = 2b + d$.

In the same way as above, we obtain $3 = (c-1)(b-d)$ from subtracting the third equation from the second one, and $3 = (1-a)(b-d)$ from subtracting the first equation from the fourth one. So $c - 1 = 1 - a$, or equivalently, $a + c = 2$.

Now we add the first two equations. This gives

$$8 = ab + c + d + bc + d + a = b(a + c) + (a + c) + 2d = 2b + 2 + 2d = 5 + d,$$

hence $d = 3$. Since $2b + d = 3$, it immediately follows that $b = 0$. Now the first equation becomes $0 + c + 3 = 3$, from which we deduce that $c = 0$. Substituting this in the second equation gives $0 + 3 + a = 5$, hence $a = 2$. Hence the only possible solution is $(a, b, c, d) = (2, 0, 0, 3)$, and by substituting this in the equations, we see that this 4-tuple is indeed a solution of the system of equations. $\square$

2. Suppose that $\frac{4n-2}{n+5}$ is the square of a rational number. Then we can write this number as $\frac{p^2}{q^2}$, where p, q are non-negative integers, $q \neq 0$, and $\gcd(p, q) = 1$. As $\gcd(p, q) = 1$, we also have $\gcd(p^2, q^2) = 1$, so there exists a non-zero integer $c \neq 0$ such that $4n - 2 = cp^2$ and $n + 5 = cq^2$. This implies that

$$22 = 4(n + 5) - (4n - 2) = 4cq^2 - cp^2 = c\big((2q)^2 - p^2\big) = c(2q - p)(2q + p).$$

Hence c is a divisor of 22. Note that 2 divides 22, so the right hand side is divisible by 2. Also note that $2q - p$ contains a factor 2 if and only if $2q + p$ does; their difference $2p$ is even. As 22 contains exactly one factor 2, if follows that $2q - p$ and $2q + p$ cannot both be even. Hence they must be odd, and c must be even.

As $p \geq 0$ and $q \geq 1$, we have $2q + p \geq 2$, which implies that the factor 11 of 22 must be factor of $2q + p$. We deduce that there are only two possibilities.

- $c = 2$, $2q + p = 11$, $2q - p = 1$;
- $c = -2$, $2q + p = 11$, $2q - p = -1$.

In the first case, we find $2p = 11 - 1 = 10$, so $p = 5$ and $q = 3$. Hence $4n-2 = 50$, so $n = 13$. Indeed, we see that $\frac{4n-2}{n+5} = \frac{50}{18} = \frac{25}{9} = \frac{5^2}{3^2}$, so $n = 13$ has the desired property. In the second case, we find $2p = 11 - (-1) = 12$, so $p = 6$, but then q is not an integer. Hence this case cannot occur. We conclude that $n = 13$ is the unique integer having the desired property. □

3. We consider the configuration in which D lies in the interior of $\triangle ABC$; the proof for the other configuration is analogous. By the inscribed angle theorem for Γ_1, we have $\angle DAC = \angle DBA$. By the inscribed angle theorem for Γ_2, we have $\angle DAB = \angle DCA$. Hence $\triangle ABD \sim \triangle CAD$ (AA), from which we deduce that

$$\frac{|BD|}{|AD|} = \frac{|AD|}{|CD|}, \tag{1}$$

or equivalently, $|AD|^2 = |BD||CD|$.

The inscribed angle theorem now gives $\angle ECB = \angle EAB = \angle DAB = \angle DCA$. Hence $\angle ECD = \angle ECB + \angle BCD = \angle DCA + \angle BCD = \angle BCA$. Now $\angle DEC = \angle AEC = \angle ABC$ by the inscribed angle theorem. Hence $\triangle CDE \sim \triangle CAB$ (AA). Analogously, $\triangle BDE \sim \triangle BAC$. We deduce that $\triangle CDE \sim \triangle EDB$, from which follows that $\frac{|CD|}{|DE|} = \frac{|ED|}{|DB|}$, or equivalently, $|DE|^2 = |BD||CD|$. Combining this with (1), we obtain $|DE|^2 = |AD|^2$, so $|DE| = |AD|$, from which we deduce that D is the midpoint of AE. □

4. We show that the minimal value of N is $2n - 1$. First, we construct an example by induction.

For $n = 3$ we can cover a 3×3-board by 5 rectangles, by putting a 1×1-rectangle in the centre, and covering the remaining squares with four 1×2- and 2×1-rectangles. Note that we used 3 rectangles of type I and 3 of type II, as required.

Now let $k \geq 3$, and assume that we can cover a $k \times k$-board with $2k - 1$ rectangles in a way satisfying all the conditions. Consider a $(k+1) \times (k+1)$-board, and cover the lower right $k \times k$ square with $2k - 1$ rectangles in a way satisfying all the conditions. Cover the top row with a $1 \times (k + 1)$-rectangle, and the remaining squares with a $k \times 1$-rectangle. We used one more rectangle of type I, and one more of type II, so the covering obtained still satisfies the conditions. Moreover, we used $2k - 1 + 2 = 2(k + 1) - 1$ rectangles.

Hence we can cover a $n \times n$-board by $2n-1$ rectangles. We now show that any covering with N rectangles must have $N \geq 2n-1$. So fix a covering of the $n \times n$-board satisfying the conditions, and let N be the number of rectangles used. Let k be the number of type I rectangles that are not of type II. Then the number of type II rectangles that are not of type I is also k. Furthermore, let l be the number of 1×1-rectangles. Then $N = 2k + l$. If $k \geq n$, then $N \geq 2n$, in which case there is nothing to prove. Hence assume that $k < n$. It then suffices to show $l \geq 2n - 2k - 1$.

Every rectangle of type I can only cover squares in a single row. Hence there are $n-k$ rows in which no square is covered by a rectangle that is of type I, but not of type II. Similarly, every rectangle of type II can only cover squares in a single column. Hence there are $n-k$ columns in which no square is covered by a rectangle that is of type II, but not of type I. Consider the $(n-k)^2$ squares that lie on such a row and such a column. They must be covered by 1×1-rectangles, hence $l \geq (n-k)^2$.

Since $(k-n+1)^2 \geq 0$, we have $k^2 + n^2 + 1 - 2kn + 2k - 2n \geq 0$, so $n^2 - 2kn + k^2 \geq 2n - 2k - 1$. Hence $l \geq (n-k)^2 \geq 2n - 2k - 1$, as desired. We conclude that the minimal value of N is $2n-1$. $\square$

5. Note that a, b, and c are positive, so the conditions of AM-GM are satisfied in each of the cases below.

First, apply AM-GM to a^2 and 1 to get

$$a^2 + 1 \geq 2\sqrt{a^2} = 2a.$$

We repeat this for b and c, and add the resulting inequalities. This yields

$$a^2 + b^2 + c^2 + 3 \geq 2a + 2b + 2c. \tag{2}$$

Next, we apply AM-GM to bc, ca, and ab, yielding

$$bc + ca + ab \geq 3\sqrt[3]{a^2b^2c^2} = 3, \tag{3}$$

using $abc = 1$. Moreover, again by AM-GM on a^2, b^2, and c^2, we have

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2} = 3. \tag{4}$$

Now we add 2 times (2), 4 times (3), and 1 time (4) to deduce that

$$2a^2 + 2b^2 + 2c^2 + 6 + 4bc + 4ca + 4ab + a^2 + b^2 + c^2 \geq 4a + 4b + 4c + 12 + 3.$$

Adding $2bc + 2ca + 2ab - 6$ to this inequality then gives

$$3a^2 + 3b^2 + 3c^2 + 6bc + 6ca + 6ab \geq 2bc + 2ca + 2ab + 4a + 4b + 4c + 9.$$

Note that the left hand side is equal to $3(a^2 + b^2 + c^2 + 2bc + 2ca + 2ab) = 3(a + b + c)^2$. For the right hand side, note that $9 = 8 + abc$. Hence the right hand side is equal to $(a+2)(b+2)(c+2)$. Both sides are positive, so dividing by 3 and taking square roots gives us

$$a + b + c \geq \sqrt{\tfrac{1}{3}(a + 2)(b + 2)(c + 2)}.$$

$\square$

IMO Team Selection Test 2, June 2013

Problems

1. Show that
$$\sum_{n=0}^{2013} \frac{4026!}{\big(n!(2013-n)!\big)^2}$$
is the square of an integer.

2. Let P be the intersection of the diagonals of a convex quadrilateral $ABCD$. Let X, Y, and Z be points on the interior of AB, BC, and CD, respectively, such that
$$\tfrac{|AX|}{|XB|} = \tfrac{|BY|}{|YC|} = \tfrac{|CZ|}{|ZD|} = 2.$$
Suppose moreover that XY is tangent to the circumcircle of $\triangle CYZ$ and that YZ is tangent to the circumcircle of $\triangle BXY$. Show that $\angle APD = \angle XYZ$.

3. Fix a sequence $a_1, a_2, a_3, \ldots$ of integers satisfying the following condition: for all prime numbers p and all positive integers k, we have
$$a_{pk+1} = pa_k - 3a_p + 13.$$
Determine all possible values of a_{2013}.

4. Determine all positive integers $n \geq 2$ satisfying
$$i + j \equiv \tbinom{n}{i} + \tbinom{n}{j} \pmod 2$$
for all i and j such that $0 \leq i \leq j \leq n$.

5. Let $ABCDEF$ be a cyclic hexagon satisfying $AB \perp BD$ and $|BC| = |EF|$. Let P be the intersection of BC and AD, and let Q be the intersection of EF and AD. Assume that P and Q are on the same side of D, and that A is on the opposite side. Let S be the midpoint of AD. Let K and L be the centres of the incircles of $\triangle BPS$ and $\triangle EQS$, respectively. Prove that $\angle KDL = 90°$.

Solutions

1. We prove the following more general statement.

$$\sum_{n=0}^{m} \frac{(2m)!}{\big(n!(m-n)!\big)^2} = \binom{2m}{m}^2. \tag{5}$$

We have

$$\frac{(2m)!}{\big(n!(m-n)!\big)^2} = \frac{(m!)^2}{\big(n!(m-n)!\big)^2} \cdot \frac{(2m)!}{(m!)^2} = \binom{m}{n}^2 \cdot \binom{2m}{m}.$$

Hence it suffices to show that

$$\sum_{n=0}^{m} \binom{m}{n}^2 = \binom{2m}{m}.$$

We will do this combinatorially. Consider $2m$ balls, numbered from 1 up to $2m$. Balls 1 up to m are coloured blue, and balls $m+1$ up to $2m$ are coloured red. We can choose m balls from these $2m$ balls in $\binom{2m}{m}$ ways. On the other hand, we can also first choose n blue balls, with $0 \leq n \leq m$, and then choose $m-n$ red balls. Equivalently, we can choose n blue balls to include, and n red balls to not include. Hence the number of ways in which one can choose m balls is also equal to

$$\sum_{n=0}^{m} \binom{m}{n}^2.$$

Hence this sum is equal to $\binom{2m}{n}$. This proves (5). □

2. By the inscribed angle theorem, we have $\angle CZY = \angle BYX$ and $\angle BXY = \angle CYZ$. It follows from this that

$$\angle XYZ = 180^\circ - \angle BYX - \angle CYZ = 180^\circ - \angle BYX - \angle BXY = \angle ABC.$$

Moreover, we see that $\triangle XBY \sim \triangle YCZ$ (AA). This implies that

$$\tfrac{|XB|}{|BY|} = \tfrac{|YC|}{|CZ|}.$$

From the assumptions, it immediately follows that $|XB| = \frac{1}{3}|AB|$, $|BY| = \frac{2}{3}|BC|$, $|YC| = \frac{1}{3}|BC|$, and $|CZ| = \frac{2}{3}|CD|$. Substituting this yields

$$\frac{\frac{1}{3}|AB|}{\frac{2}{3}|BC|} = \frac{\frac{1}{3}|BC|}{\frac{2}{3}|CD|},$$

hence $\frac{|AB|}{|BC|} = \frac{|BC|}{|CD|}$. As $\triangle XBY \sim \triangle YCZ$, we also have $\angle ABC = \angle XBY = \angle YCZ = \angle BCD$. Hence $\triangle ABC \sim \triangle BCD$ (SAS). We deduce that $\angle CAB = \angle DBC$. From this, it follows that

$$\angle PAB + \angle ABP = \angle CAB + \angle ABD = \angle DBC + \angle ABD = \angle ABC.$$

We established before that $\angle ABC = \angle XYZ$, so by the external angle theorem applied to $\triangle ABP$, we have

$$\angle BPC = \angle PAB + \angle ABP = \angle ABC = \angle XYZ.$$

As $\angle BPC = \angle APD$, it follows that $\angle APD = \angle XYZ$. $\square$

3. Let q and t be primes. Substituting $k = q$ and $p = t$ yields

$$a_{qt+1} = ta_q - 3a_t + 13.$$

Substituting $k = t$ and $p = q$ yields

$$a_{qt+1} = qa_t - 3a_q + 13.$$

Hence the right hand sides are equal, so

$$ta_q - 3a_t = qa_t - 3a_q,$$

or equivalently,

$$(t+3)a_q = (q+3)a_t.$$

In particular, we have $5a_3 = 6a_2$ and $5a_7 = 10a_2$. Substituting $k = 3$ and $p = 2$ now gives

$$a_7 = 2a_3 - 3a_2 + 13 = 2 \cdot \frac{6}{5}a_2 - 3a_2 + 13.$$

As $a_7 = \frac{10}{5}a_2$, this implies that

$$\frac{13}{5}a_2 = 13,$$

hence $a_2 = 5$. We deduce that for all primes p, we have $a_p = \frac{(p+3)a_2}{5} = p + 3$.

Substituting $k = 4$ and $p = 3$ gives

$$a_{13} = 3a_4 - 3a_3 + 13.$$

As $a_{13} = 16$ and $a_3 = 6$, it follows that $3a_4 = 21$, or equivalently, $a_4 = 7$.

Finally, substitute $k = 4$ and $p = 503$ to get

$$a_{2013} = a_{4\cdot 503+1} = 503\cdot a_4 - 3a_{503} + 13 = 503\cdot 7 - 3\cdot(503+3) + 13 = 2016.$$

Hence $a_{2013} = 2016$, so this is the only possible value.

It remains to check that this value is attained for some sequence of integers, i.e. there exists a sequence satisfying the condition. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = n + 3$ for all $n \geq 1$. Then for all primes p and all positive integers k, we have

$$a_{pk+1} = pk+4 = pk+3p-3p+4 = (pk+3p)-3p-9+9+4 = pa_k - 3a_p + 13,$$

as desired. $\square$

4. We first show that n satisfies the condition if and only if $\binom{n}{i} \equiv i+1 \mod 2$ for all i such that $0 \leq i \leq n$. Suppose that $\binom{n}{i} \equiv i+1 \mod 2$ for all i, then we have $\binom{n}{i} + \binom{n}{j} \equiv i+1+j+1 \equiv i+j \mod 2$ for all i and j. Hence n satisfies the condition. Conversely, suppose that n satisfies the condition. As $\binom{n}{0} = 1$, we have $i \equiv 1 + \binom{n}{i} \mod 2$ for all i, so $\binom{n}{i} \equiv i - 1 \equiv i + 1 \mod 2$ for all i.

Write n as $n = 2^k + m$ with $0 \leq m < 2^k$. Since $n \geq 2$, we may assume that $k \geq 1$. Consider

$$\binom{n}{2^k-2} = \binom{2^k+m}{2^k-2} = \frac{(2^k+m)(2^k+m-1)\cdots(m+4)(m+3)}{(2^k-2)(2^k-3)\cdots 2\cdot 1}.$$

The product in the denominator has $\lfloor\frac{2^k-2}{2}\rfloor$ factors divisible by 2, $\lfloor\frac{2^k-2}{4}\rfloor$ factors divisible by 4, $\ldots$, $\lfloor\frac{2^k-2}{2^{k-1}}\rfloor$ factors divisible by 2^{k-1} and no factors divisible by 2^k. The product in the numerator consists of 2^k-2 consecutive factors, hence has at least $\lfloor\frac{2^k-2}{2}\rfloor$ factors divisible by 2, at least $\lfloor\frac{2^k-2}{4}\rfloor$ factors divisible by 4, $\ldots$, at least $\lfloor\frac{2^k-2}{2^{k-1}}\rfloor$ factors divisible by 2^{k-1}. We deduce that the number of factors 2 of the product in the numerator is at least that of the one in the denominator. If 2^k occurs as factor of the numerator, then the number of factors 2 of the product in the numerator is greater than that of the one in the denominator. This is the case if $m+3 \leq 2^k$, i.e. if $m \leq 2^k - 3$. So if $m \leq 2^k - 3$, then $\binom{n}{2^k-2}$ is even, whereas $2^k - 2$ is even as well. Hence $n = 2^k + m$ does not satisfy the condition.

We deduce that n can only satisfy the condition if there exists a $k \geq 2$ such that $2^k - 2 \leq n \leq 2^k - 1$. If n is odd, then $\binom{n}{0} + \binom{n}{1} = 1 + n \equiv 0 \mod 2$, so n does not satisfy the condition. This implies that n can only satisfy the condition if n is of the form $n = 2^k - 2$ with $k \geq 2$ is. Suppose that n is of that form. We show that n satisfies the condition. We have the following.

$$\binom{2^k - 2}{c} = \frac{(2^k-2)(2^k-3)\cdots(2^k-c)(2^k-c-1)}{c\cdot(c-1)\cdots 2\cdot 1}.$$

Note that the number of factors 2 in $2^k - i$ is equal to the number of factors 2 in i for $1 \leq i \leq 2^k - 1$. Hence the number of factors 2 in the numerator is equal to the number of factors 2 in the product $(c+1)\cdot c\cdot(c-1)\cdots 2$, or equivalently, the number of factors 2 in the denominator plus the number of factors 2 in $c+1$. We deduce that $\binom{2^k-2}{c}$ is even if and only if c is odd. Hence $n = 2^k - 2$ satisfies the condition.

We conclude that n satisfies the condition if and only if n is of the form $2^k - 2$ with $k \geq 2$. $\square$

5. The configuration is fixed by the order in which the points A up to F occur on the circle, and by the condition on the points P and Q on AD. So there is no need to distinguish between different configurations. (The positions of P and Q with respect to one another are irrelevant.)

First note that S is the centre of the circumcircle of $ABCDEF$, as AD is its diameter, since $\angle ABD = 90°$, and S is the midpoint of AD. We will show that $\angle KDS = \angle KBS$. As KS is the angle bisector of $\angle BSD$, we have $\angle BSK = \angle KSD$. Furthermore, $|SD| = |SB|$, as the segments SD and SB are both radii of the circumcircle of $ABCDEF$. Since $|SK| = |SK|$, we have $\triangle DSK \cong \triangle BSK$ (SAS). Hence $\angle KDS = \angle KBS$.

As BK is the angle bisector of $\angle CBS$, we have $\angle KBS = \angle CBK = \frac{1}{2}\angle CBS$. Hence $\angle KDS = \frac{1}{2}\angle CBS$.

Analogously, we find $\angle LDS = \frac{1}{2}\angle QES$, which implies that $\angle LDS = \frac{1}{2}\cdot(180° - \angle FES) = 90° - \frac{1}{2}\angle FES$. Hence $\angle LDS + \angle KDS = 90° - \frac{1}{2}\angle FES + \frac{1}{2}\angle CBS$. Now note that $\triangle SBC \cong \triangle SEF$ (SSS), so $\angle FES = \angle CBS$. We conclude that $\angle KDL = \angle LDS + \angle KDS = 90°$. $\square$

Junior Mathematical Olympiad, October 2012

Problems

Part 1

1. A bag contains red and blue marbles. Of these marbles $\frac{3}{5}$ is blue, the rest is red.
If we double the number of red marbles in the bag, what portion of the marbles will be blue?

A) $\frac{1}{5}$ B) $\frac{3}{7}$ C) $\frac{4}{7}$ D) $\frac{3}{5}$ E) $\frac{4}{5}$

2. Alyssa sums odd numbers. First she takes 1, then she calculates $1+3$, next she calculates $1+3+5$, then $1+3+5+7$, etc. In this way she obtains a very long list of numbers. The largest number that she calculates is $1+3+5+7+\cdots+197+199$.
How many of the numbers on Alyssa's list end with a 4?

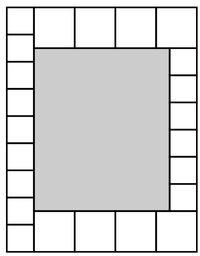
A) 9 B) 10 C) 19 D) 20 E) 50

3. We have got a dice and we will colour its pips. For each of the six faces we choose one of the colours red, white and blue and colour all pips on that face in that colour. There is one rule: in each vertex we must choose three different colours for the three adjacent faces.
In how many ways can we colour the dice?

A) 3 B) 6 C) 12 D) 27 E) 36

4. The grey rectangle in the figure has width 8. Around it is a ring consisting of squares of two different sizes; see the figure.
What is the height of the grey rectangle?

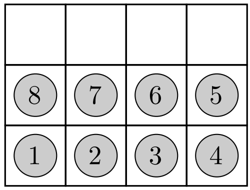
A) 9 B) $\frac{28}{3}$ C) $\frac{19}{2}$ D) $\frac{48}{5}$ E) 10



5. Ans, Ben, Carla and Dirk participate to a lottery. In a hat there are eight cards with the numbers 1 to 8. One by one each person takes two cards out of the hat such that Dirk takes the last two cards. To make it even more exciting, everybody adds his or her two numbers and tells the result to the others. For Ans this is 10, for Ben 14 and for Carla 5.
What are the numbers that are on Dirk's cards?

A) 1 and 6 B) 1 and 7 C) 2 and 5 D) 2 and 6 E) 3 and 4

6. On a 3×4-board there are eight pieces with the numbers 1 to 8, see the figure. Pieces may be captured horizontally and vertically. I.e.: if a piece in the horizontal or vertical direction borders on an empty square on the one side and on an occupied square on the other side, then the piece on the occupied square may jump to the empty square. The piece on the middle square is captured and removed from the board. Jan keeps capturing pieces until there is only one piece left on the board.
The only possibilities for this last piece are:



A) 1 and 2 B) 3 and 4 C) 2 and 3 D) 1 and 4 E) 1, 2, 3 and 4

7. How many zeros does the number that is the result of the division

$$1010101010101010101 : 101$$

contain?

A) 8 B) 9 C) 10 D) 11 E) 12

8. Jan and Katrijn live on a straight dike, exactly 11,7 kilometres from each other. Jan cycles to Katrijn. On each place on the road he can see how far he is from his own house and how far is he from Katrijn's house. Both distances we round to whole kilometres. On some part of the dike this rounded distances are equal.
How many kilometres long is this part?

A) 0,1 B) 0,3 C) 0,5 D) 0,7 E) 0,9

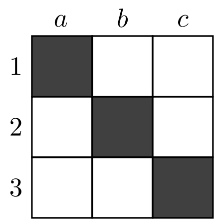
9. Anne draws a triangle of which the sides have have different lengths and the shortest side is 6 cm long. Bert redraws Anne's triangle, but he draws it twice as big (all sides twice as long). Christa also redraws Anne's triangle, but she draws it three times as big. It appears that Anne's triangle has a side that has the same length as one of the sides of Bert's triangle and that Bert's triangle has a side that has the same length as one of the sides of Christa's triangle.
How many centimetres long is the longest side of Anne's triangle?

A) 12 B) 18 C) 24 D) 30 E) 36

10. We calculate a *recurrent average* of the numbers 1, 2, 3, 4 and 5 in the following way. First take the average of two of these numbers. Take the average of this result and a third numbers. Then take the average of this next result with a fourth number. Finally take the average of this result with the last number.
What is the difference between the largest and smallest result we may obtain in this way?

A) 0 B) $\frac{3}{2}$ C) $\frac{31}{16}$ D) $\frac{17}{8}$ E) $\frac{65}{16}$

11. Of a 3×3-board with squares $a1$ till $c3$ three squares are painted black. This is done in such a way that no two black fields are neighbouring fields (i.e. sharing an edge). In the figure there is one way to do this.
In how many ways can this be done?

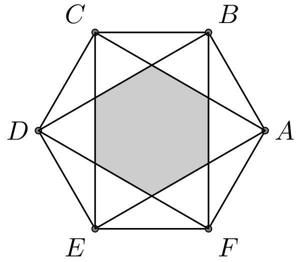


A) 14 B) 18 C) 20 D) 22 E) 30

12. While Snow White is cooking, the seven dwarfs play chess. Doc plays against all other six dwarfs. Happy plays against five others, Grumpy against four others, Sneezy against three others, Bashful again two others and Sleepy plays against one other dwarf, after which he falls asleep.
Against how many dwarfs does Dopey, the seventh dwarf, play?

A) 0 B) 1 C) 3 D) 5 E) 6

13. In a regular hexagon $ABCDEF$ the diagonals are drawn as in the figure. The area of the hexagon $ABCDEF$ is 1.
What is the area of the grey hexagon?



A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{3}{8}$ E) $\frac{5}{16}$

14. Peter has some numbers of which the average is 20. If he leaves out the smallest number, the average of the other numbers will be 22. If he leaves out the biggest number, then the average of the others is 13. If he leaves out both the smallest and the biggest number, then the average becomes 14.
How many numbers does Peter have?

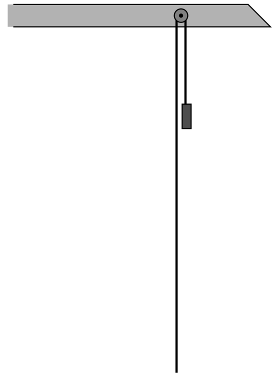
A) 4 B) 5 C) 6 D) 7 E) 8

15. Ann, Bo, Cas, Dex and Eva cycle in the same direction on a long road, each with a constant speed. Their speeds are different and for each pair there is a moment when they cycle exactly next to each other and the one takes over the other. After the trip Ann tells: "First I took over Bob, then I got taken over by Cas. Then I took over Dex and finally I got taken over by Eva."
Bo tells: "I got taken over by Eva. That was after the moment when I cycled next to Ann, but before the moment I cycled next to Cas or Dex."
Put the five in order from fastest to slowest.

A) Eva, Cas, Ann, Bo, Dex
B) Cas, Eva, Ann, Bo, Dex
C) Eva, Cas, Ann, Dex, Bo
D) Cas, Eva, Ann, Dex, Bo
E) This cannot be determined

Part 2

The answer to each problem is a number.



1. A long chain hangs over the pulley of a crane, see the figure. On one side of the chain a weight is attached. The length of the part of the chain between the pulley and the weight is exactly one fourth of the length of the chain between the pulley and the other end (the size of the pulley itself is negligible). The chain is hanging exactly in balance. The chain and the weight together weigh 320 kilogram.
How many kilograms does the chain weigh?

2. Huey, Dewey and Louie are going to run. They start at the same time and each of them walks at constant speed. When Huey finished three laps, Dewey is exactly halfway his third lap. When Dewey finished his third lap, Louie is exactly halfway his third lap. After a while Huey, Dewey and Louie are together at the start at the same moment.
How many laps did Louie finish at this moment?

3. What is the largest number of $2\times2\times1$-bricks that one can put in a box of $7\times7\times6$? The bricks must be aligned parallel to the sides of the box.

4. Anne is completely broke, but today (day 1) she got a new job. She works every day (even during the weekends) and after every shift, she is always paid 13 euro. After every shift, she spends the money on groceries. On the first day Anne spends 1 euro on groceries, on the second day 2 euro, on the third day 3 euro, etc.
What is the first day on which Anne cannot afford her groceries anymore?

5. Ronald has bought a large barrel of water. The first day of the year, 1 January 2012, he uses half of it. The second day he uses a third of the remaining water, and on the third day a fourth of the then remaining water, and so on until the 365th day of the year. On the last day of the year (2012 has 366 days) he has only one litre of water left and he drinks it.
How many litres of water did the barrel contain when Ronald bought it?

6. To a three digit number you add its three digits. For example, the number 216 becomes $216 + 2 + 1 + 6 = 255$.
What is the largest three digit number that you cannot make in this way?

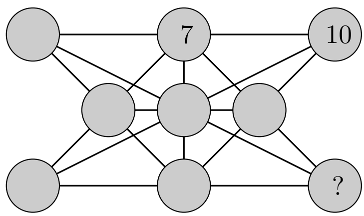
7. Four students made a multiple choice test consisting of four questions. For each question they could choose from the answers A, B, C, D and E. The first student answered DDAE, the second CBAD, the third CDAC and the fourth answered BBCC. Unfortunately each of them had only two questions right.
What are the four right answers?

8. To a two digit number we apply the following recipe:

- multiply the number with itself,
- then subtract 6 times the initial number from this result and
- finally add 9 to that result.

For example with 41 you get: $41\times41 = 1681$, $1681 - 6\times41 = 1681 - 246 = 1435$, $1435 + 9 = 1444$. If the result is equal to the initial number, but then with the two digits interchanged, followed by two zeros, then we call the initial number *great*. For example, if the result for 41 would have been equal to 1400, then 41 would have been great. There is exactly one great two digit number.
What is that number?

9. In each of the nine circles a number must be placed. It must be done in such a way that when you add the three numbers on a line, you would get the same answer for each line. In two circles a number already has been placed.
What is the number that must be put on the place of the question mark?



10. A mathematical landscape gardener makes a geometrical design for a big garden. On the big land he places two poles at a distance of 20 metres from each other. On each place where the distance to each of the poles is an integer number of metres that is at most 14, he plants a box-tree.
How many box-trees does he plant?

Solutions

Part 1

1.	B) $\frac{3}{7}$	**6.**	E) 1, 2, 3 and 4	**11.**	D) 22
2.	D) 20	**7.**	E) 12	**12.**	C) 3
3.	B) 6	**8.**	D) 0,7	**13.**	B) $\frac{1}{3}$
4.	D) $\frac{48}{5}$	**9.**	A) 12	**14.**	D) 7
5.	C) 2 and 5	**10.**	D) $\frac{17}{8}$	**15.**	B) C, E, A, B, D

Part 2

1.	200	**6.**	995
2.	25	**7.**	DBAC
3.	72	**8.**	63
4.	26	**9.**	10
5.	366	**10.**	81

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