# Final round
# Dutch Mathematical Olympiad

Friday 15 September 2017
Technische Universiteit Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. Points can also be awarded to partial solutions.
- Write down all the steps of your argumentation. A clear reasoning is just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem and also hand in your draft sheets (for each problem separately!). Good luck!

**1. Version for klas 5 & klas 4 and below**

We consider positive integers written down in the (usual) decimal system. Within such an integer, we number the positions of the digits from left to right, so the leftmost digit (which is never a 0) is at position 1.

An integer is called *even-steven* if each digit at an *even* position (if there is one) is greater than or equal to its neighbouring digits (if these exist).

An integer is called *oddball* if each digit at an *odd* position is greater than or equal to its neighbouring digits (if these exist).

For example, 3122 is oddball but not even-steven, 7 is both even-steven and oddball, and 123 is neither even-steven nor oddball.

(a) Prove: every even-steven integer greater than 9 can be obtained by adding two oddball integers.

(b) Prove: there exists an oddball integer greater than 9 that cannot be obtained by adding two even-steven integers.

**1. Version for klas 6**

We consider positive integers written down in the (usual) decimal system. Within such an integer, we number the positions of the digits from left to right, so the leftmost digit (which is never a 0) is at position 1.

An integer is called *even-steven* if each digit at an *even* position (if there is one) is greater than or equal to its neighbouring digits (if these exist).

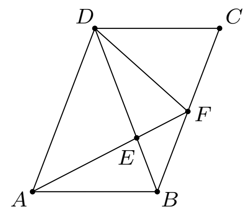
An integer is called *oddball* if each digit at an *odd* position is greater than or equal to its neighbouring digits (if these exist).

For example, 3122 is oddball but not even-steven, 7 is both even-steven and oddball, and 123 is neither even-steven nor oddball.

(a) Prove: every oddball integer greater than 9 can be obtained by adding two oddball integers.

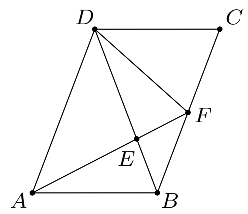
(b) Prove: there exists an oddball integer greater than 9 that cannot be obtained by adding two even-steven integers.

**2. Version for klas 4 & below**




A parallelogram $ABCD$ with $|AD| = |BD|$ has been given. A point $E$ lies on line segment $BD$ in such a way that $|AE| = |DE|$. The (extended) line $AE$ intersects line segment $BC$ in $F$.
Prove that $\angle CDF = \angle ADB$.

**2. Version for klas 5 & klas 6**




A parallelogram $ABCD$ with $|AD| = |BD|$ has been given. A point $E$ lies on line segment $BD$ in such a way that $|AE| = |DE|$. The (extended) line $AE$ intersects line segment $BC$ in $F$. Line $DF$ is the angle bisector of angle $CDE$.
Determine the size of angle $ABD$.

**3.** Six teams participate in a hockey tournament. Each team plays exactly once against each other team. A team is awarded 3 points for each game they win, 1 point for each draw, and 0 points for each game they lose. After the tournament, a ranking is made. There are no ties in the list. Moreover, it turns out that each team (except the very last team) has exactly 2 points more than the team ranking one place lower.
Prove that the team that finished fourth won exactly two games.

**4. Version for klas 5 & klas 4 and below**

If we divide the number 13 by the three numbers 5, 7, and 9, then these divisions leave remainders: when dividing by 5 the remainder is 3, when dividing by 7 the remainder is 6, and when dividing by 9 the remainder is 4. If we add these remainders, we obtain $3 + 6 + 4 = 13$, the original number.

(a) Let $n$ be a positive integer and let $a$ and $b$ be two positive integers smaller than $n$. Prove: if you divide $n$ by $a$ and $b$, then the sum of the two remainders never equals $n$.

(b) We consider integers $n > 229$ having the following property: if you divide $n$ by 99, 132, and 229, then the sum of the three remainders is $n$.
Prove that for such an integer $n$ the two remainders obtained when dividing $n$ by 99 and 132 add up to 229.

(c) Determine all integers $n > 229$ having the property that if you divide $n$ by 99, 132, and 229, the sum of the three remainders is $n$.

**4. Version for klas 6**

If we divide the number 13 by the three numbers 5, 7, and 9, then these divisions leave remainders: when dividing by 5 the remainder is 3, when dividing by 7 the remainder is 6, and when dividing by 9 the remainder is 4. If we add these remainders, we obtain $3 + 6 + 4 = 13$, the original number.

(a) Let $n$ be a positive integer and let $a$ and $b$ be two positive integers smaller than $n$. Prove: if you divide $n$ by $a$ and $b$, then the sum of the two remainders never equals $n$.

(b) Determine all integers $n > 229$ having the property that if you divide $n$ by 99, 132, and 229, the sum of the three remainders is $n$.

**5.** The eight points below are the vertices and the midpoints of the sides of a square. We would like to draw a number of circles through the points, in such a way that each pair of points lie on (at least) one of the circles.
Determine the smallest number of circles needed to do this.