Second round Dutch Mathematical Olympiad

Friday 16 March 2018

Solutions

B-problems

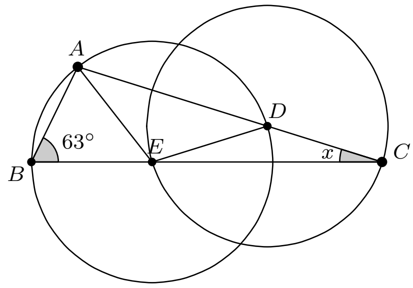
1. A, D, B, C Let a, b, c, and d be the scores of Anouk, Bart, Celine, and Daan, respectively. We are given that

$$b + d = a + c, \qquad a + b > c + d \quad \text{and} \quad d > b + c.$$

By adding a to both sides of the equation on the left, we obtain $a + b + d = 2a + c$. By adding d to both sides of the inequality in the middle, we obtain $a + b + d > c + 2d$. By combining the two expressions, we get $2a + c > c + 2d$. This shows that $a > d$.

We know that $a > d > b + c$, hence Anouk has the highest score and Daan the second highest score. From the equation $b + d = a + c$ and $a > d$ it follows that $b + d > c + d$, and hence that $b > c$. We conclude that the order of the students is: Anouk, Daan, Bart, Celine.

2. 18 degrees Let x denote the size of the angle at vertex C. Since D is the centre of a circle through C and E, the points C and E have the same distance to D. That is, triangle CDE is isosceles with apex D. It follows that $\angle DEC = x$ and $\angle CDE = 180° - 2x$, because the angles of a triangle sum to 180°.



Since $\angle CDE$ and $\angle ADE$ together form a straight angle, we can deduce that $\angle ADE = 2x$. Triangle AED is isosceles with the apex being E. Hence, $\angle DAE = 2x$. Triangle ABE is also an isosceles triangle with apex E. It follows that $\angle EAB = \angle ABE = 63°$.

Since the sum of the angles in triangle ABC sum to 180 degrees, we obtain $63° + (63° + 2x) + x = 180°$. This implies that $3x = 180° - 126° = 54°$. The requested angle x is therefore $\frac{1}{3} \cdot 54° = 18°$.

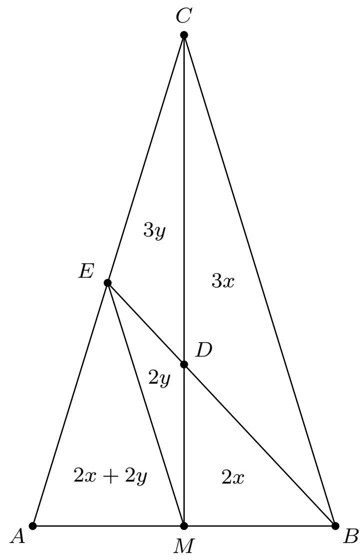
3. 2442 Suppose that both n and $n - 2018$ are palindromic numbers. Let a be the first (and hence also the last) digit of n. Observe that the last digit of $n - 2018$ is equal to $a - 8$ if $a \geqslant 8$ and to $a - 8 + 10 = a + 2$ if $a < 8$.

We start by considering the case that n is five or more digits long. If $a \geqslant 2$, then the first digit of $n - 2018$ is equal to a or $a - 1$. The last digit, however, is equal to $a - 8$ or $a + 2$. It follows that $n - 2018$ is not palindromic. Now consider the case that $a = 1$. If n is five digits long, the number $n - 2018$ lies in the range from $10000 - 2018 = 7982$ to $19999 - 2018 = 17981$. Hence, the first digit of $n - 2018$ is equal to 7, 8, 9, or 1. If n is six or more digits long, the first digit of $n - 2018$ equals 1 or 9 (if for instance $100000 \leqslant n \leqslant 102017$). In both cases, however, the last digit of $n - 2018$ is equal to $a + 2 = 3$. So $n - 2018$ is not palindromic.

What remains is the case that n is four digits long. Since n is greater than 2018, we have $a \geqslant 2$. If $a \geqslant 4$, then the first digit of $n - 2018$ is equal to $a - 2$ or $a - 3$. However, the last digit of $n - 2018$ is equal to $a - 8$ or $a + 2$. Hence, $n - 2018$ is not palindromic. If $a = 3$, then the first digit of $n - 2018$ is equal to 1 or 9. However, the last digit equals 5, so $n - 2018$ is not palindromic.

The only possible case is $n = 2bb2$, where $b \geqslant 1$. We see that $2112 - 2018 = 94$ is not palindromic. If $b \geqslant 2$, the last digit of $n - 2018$ equals 4 and the first digit equals b. Therefore, the only possibility is $n = 2442$. The difference $2442 - 2018 = 424$ is indeed a palindromic number.

4. $\frac{3}{4}$ The area of a triangle is equal to a half times the product of its base and its height. If we choose DM and CD as the bases of the triangles BDM and BCD, then the two triangles have equal heights. It follows that the ratio of their areas is $2:3$. Let the area of triangle BDM be $2x$. Then the area of BCD equals $3x$.



Similarly, the ratio of the areas of triangles DEM and CDE is $2:3$. Let the area of triangle DEM be $2y$. Then the area of CDE is equal to $3y$. Triangles BEM and AEM have equal areas since M is the midpoint of segment AB. It follows that the area of triangle AEM is equal to $2x+2y$.

Finally, we observe that triangles ACM and BCM have equal areas, namely half the area of triangle ABC. It follows that $(2x+2y)+2y+3y=2x+3x$, and hence $7y=3x$.

The total area of triangle ABC equals $10x$. Triangle BCE has an area of $3x+3y=\frac{30}{7}x$. The area of triangle ABE is therefore equal to $10x-\frac{30}{7}x=\frac{40}{7}x$. It follows that the ratio $\frac{|CE|}{|EA|}$ is equal to $\frac{3}{4}$.

5. 110 We consider sawtooth numbers using digits 1, 2, and 3 only. We will distinguish between *increasing* and *decreasing* sawtooth numbers. A sawtooth number is called increasing if the second digit is greater than the first, and called decreasing if the second digit is smaller than the first. There are six sawtooth numbers of length 2, namely 12, 13, 23, 21, 31, and 32. The first three are increasing and the last three are decreasing.

If we want to make an increasing sawtooth number of length 3, this can be done in two ways. We can start with digit 1 and place a decreasing sawtooth number of length 2 after it. Or, we can start with digit 2 and place a decreasing sawtooth number of length 2 after it that starts with digit 3.

In the same way, we can make increasing sawtooth numbers of length 4 by either starting with digit 1 and placing a decreasing sawtooth number of length 3 after it, or starting with digit 2 and placing a decreasing sawtooth number of length 3 after it that starts with digit 3. The increasing sawtooth numbers of length 5, 6, 7, and 8 are constructed in the same way.

For decreasing sawtooth numbers, we have a similar way of constructing them from increasing sawtooth numbers that are one digit shorter. Here, we start with digit 3 and place an increasing sawtooth number after it, or we start with digit 2 and place an increasing sawtooth number after it that starts with digit 1.

We now construct a table for the number of sawtooth numbers of lengths $2,3,\ldots,8$. By the above construction of sawtooth numbers from smaller sawtooth numbers, the table can be easily filled when working from left to right.

length	2	3	4	5	6	7	8
increasing, first digit 1	2	3	5	8	13	21	34
increasing, first digit 2	1	2	3	5	8	13	21
decreasing, first digit 2	1	2	3	5	8	13	21
decreasing, first digit 3	2	3	5	8	13	21	34

We find a total of $34+21+21+34=110$ sawtooth numbers of length 8.

C-problems

C1. A distribution of the balls that follows all three rules will be called a *correct* distribution. The box containing the larger number of balls will be called the *fullest* box. It is clear that at least $1 + 2 = 3$ balls are needed to obey the first two rules.

We first consider the case that n is **odd**.
If $n = 3$, then the fullest box must have two balls. It therefore has a value of at least $1 + 2 = 3$, while the other box has one ball and hence a value of at most 3. The third rule is broken so there is no correct distribution.

For $n = 5$ there is a correct distribution: put balls 1, 2, and 3 in one box and put balls 4 and 5 in the other box. Since $4 + 5$ is at least two more than $1 + 2 + 3$, this is indeed a correct distribution.

If there is a correct distribution for n balls, there is also one for $n + 2$ balls. Indeed, we can simply add ball $n + 1$ to the fullest box and add ball $n + 2$ to the other box. The value of the fullest box increases by less than the other box, hence we still follow rule 3.

Since we have a correct distribution for $n = 5$, we also have a correct distribution for $n = 7$. Then, we also find a correct distribution for $n = 9$, $n = 11$, et cetera.

Now we consider the case that n is **even**.
If $n = 4$, the fullest box must have at least three balls and hence has a value of at least $1 + 2 + 3 = 6$. The other box has only one ball and has a value of at most 4. This means that the third rule is broken. There is no correct distribution.

If $n = 6$, the fullest box must have at least four balls. It therefore has a value of at least $1 + 2 + 3 + 4 = 10$. The other box has at most two balls and therefore has a value of at most $5 + 6 = 11$. Since $11 < 2 + 10$, there is no correct distribution.

For $n = 8$, there is a correct distribution: put balls 1 to 5 into one box and put balls 6 to 8 in the other box. The fullest box then has a value of $1 + 2 + 3 + 4 + 5 = 15$, and the other box has a value of $6 + 7 + 8 = 21$. Since $21 \geqslant 2 + 15$, this is indeed a correct distribution.

Again, we see that a correct distribution of n balls gives a correct distribution with $n + 2$ balls by adding ball $n + 1$ to the fullest box and adding ball $n + 2$ to the other box. Since we have a correct distribution for $n = 8$, we thus find correct distributions for $n = 10, 12, 14, \ldots$.

We conclude: for $n = 1, 2, 3, 4, 6$ there is no correct distribution, but for all other positive integers n a correct distribution does exist.

C2. (a) Yes, such a number a exists. For example, take $a = 33$. Then we have: $16 + a = 7^2$, $3 + a = 6^2$, and $16 \cdot 3 + a = 9^2$.
A suitable a *can be easily found by trying* $48 + a = 49$, 64, 81.

(b) No, such a number a does not exist. The numbers $20 + a$ and $18 + a$ cannot both be squares since the difference of two squares is never equal to 2. Indeed, suppose that $m^2 - n^2 = 2$. Then we would have $(m + n)(m - n) = 2$, where $m > n$. Since 1 and 2 are the only positive divisors of 2, we would have $m + n = 2$ and $m - n = 1$. But this would imply that $2m = (m + n) + (m - n) = 3$, which is impossible since $2m$ is even whereas 3 is odd.

(c) Given an odd integer n, we will show that there exist integers a, x, y, and z such that

$$\begin{aligned} 2018 + a &= x^2, & (1)\\ n + a &= y^2, & (2)\\ 2018n + a &= z^2. & (3) \end{aligned}$$

If we subtract equation (2) from equation (1), we obtain

$$2018 - n = x^2 - y^2 = (x - y)(x + y).$$

We therefore choose x and y such that $x + y = 2018 - n$ and $x - y = 1$. Addition of these two equations gives us $2x = 2019 - n$, subtraction gives us $2y = 2017 - n$. We therefore choose

$$x = \frac{2019 - n}{2} \qquad \text{and} \qquad y = \frac{2017 - n}{2}.$$

Since n is odd, these are integers. Indeed, we now have $x - y = 1$ and $x + y = 2018 - n$, and hence that

$$2018 - n = x^2 - y^2. \tag{4}$$

We let $a = y^2 - n$. Then equation (2) certainly holds. Equation (1) holds as well, since $2018 + a = 2018 + y^2 - n = x^2$ (the second equation follows from (4)).
Finally, we show that $2018n + a$ is a square. We successively obtain

$$\begin{aligned} 2018n + a &= 2018n + y^2 - n \\ &= 2017n + \frac{(2017 - n)^2}{4} \\ &= \frac{4 \cdot 2017n + 2017^2 - 2 \cdot 2017n + n^2}{4} \\ &= \frac{2017^2 + 2 \cdot 2017n + n^2}{4} \\ &= \frac{(2017 + n)^2}{4} = \left(\frac{2017 + n}{2}\right)^2. \end{aligned}$$

Since n is odd, the number $\frac{2017+n}{2}$ is an integer. By choosing $z = \frac{2017+n}{2}$, we have found integers a, x, y, and z for which equations (1), (2), and (3) are true.