First Round
Dutch Mathematical Olympiad

22 January – 1 February 2018

Solutions

A1. B) 4 A child on a chair gives a total of 6 legs. A child on a stool gives a total of 5 legs. With 7 chairs we would already have $7 \times 6 = 42$ legs, which is too much. We make a table listing the number of chairs and the corresponding total number of legs.

Number of chairs:	0	1	2	3	**4**	5	6
Total number of legs:	0	6	12	18	**24**	30	36
Remaining legs:	39	33	27	21	**15**	9	3

The remaining number of legs must come from the children seated on stools. Hence, this number must be divisible by 5. The only number in the third row that is divisible by 5 is 15. Therefore, the number of chairs must be 4 (and the number of stools is $\frac{15}{5} = 3$).

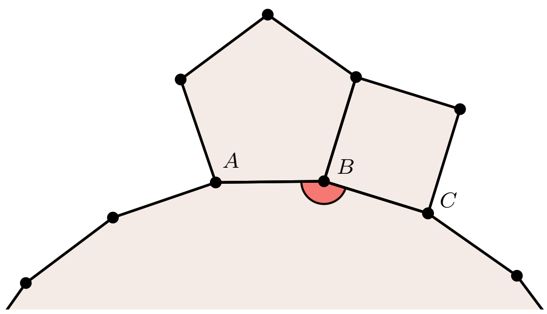
A2. C) C The statements of C and E cannot both be true because it would imply that E is a knave contradicting the fact that he told the truth. Similarly, the statements of B, C, and D cannot all be true since that would imply that D is a knave, contradicting the fact that he told the truth. The knave must therefore be C or E and, at the same time, it must be B, C, or D. Hence, C must be the knave.

Finally, we show that the situation where C is the knave is indeed consistent with the statements. Suppose that C and D are the only ones with shoe size 40. Suppose also that C, D, and E have a goldfish, and that C is the only knave. Under those circumstances, the statements by A, B, D, and E are all true, while the statement by C is false.

A3. B) 10 The angles of a regular n-gon are $180 - \frac{360}{n}$ degrees. Indeed, walking along the n-gon you will change direction n times after which you have made a complete 360 degree turn. The turn at each vertex must therefore be $\frac{360}{n}$ degrees. This means that the corresponding angle of the n-gon is $180 - \frac{360}{n}$ degrees.

A square therefore has angles equal to $180 - 90 = 90$ degrees and a regular pentagon has $180 - 72 = 108$ degree angles.

A square and a regular pentagon joined along an edge, as in the figure, determine a $360-90-108 = 162$ degree angle (angle ABC in the figure). As $162 = 180 - 18$, this is precisely the angle of a regular 20-gon since $18 = \frac{360}{20}$. Sides AB and BC have the same length. Hence, after twenty steps (10 squares and 10 pentagons), we have come full circle and the inner edges of the squares and pentagons form a regular 20-gon.



A4. D) 7 The only list of 9 numbers satisfying the first condition has each of the nine digits as a separate one-digit number. This list does not meet the second requirement (for example: 2 is divisible by 1).

A list of 8 numbers satisfying the first condition must have seven one-digit numbers and one two-digit number. However, by the second condition we cannot have more than five one-digit numbers: apart from $5, 7, 9$ we can have at most one of the numbers $1, 2, 4, 8$ and at most one of the numbers $3, 6$.

A list of 7 numbers that satisfies both conditions does exist. For example, take 5, 6, 7, 8, 9, 23, 41. The maximum length of Julian's list is therefore 7.

A5. E) 4 Let's denote the eight people that Quintijn meets at the party by A, B, C, D, E, F, G, and H (in order of increasing number of handshakes). Since there are 9 people present at the party, the number of people someone can shake hands with is always a number from 0 to 8. If someone does 8 handshakes, they must shake everyone's hand which implies that nobody can do 0 handshakes. The answers Quintijn gets must therefore be either the numbers 0 to 7 or the numbers 1 to 8. We consider both cases separately.

- Suppose that A, B, ..., H shake $0, 1, \ldots, 7$ hands.
  Person H does 7 handshakes and must therefore shake hands with everyone except A. Person B only shakes hands with H.

  Person G does 6 handshakes and must therefore shake hands with everyone except A and B. Now person C has already used up two handshakes (with H and G) and cannot shake anyone else's hand.

  Person F does 5 handshakes and must therefore shake hands with everyone except A, B, and C. Now person D has already used up 3 handshakes (with H, G, and F) and cannot shake anyone else's hand.

  Person E does 4 handshakes and must therefore shake hands with everyone except A, B, C, and D.

  We see that Quintijn shakes hands with exactly four people: H, G, F, and E.
- Suppose that A, B, ..., H shake $1, 2, \ldots, 8$ hands.
  Person H does 8 handshakes and must therefore shake hands with everyone. Person A only shakes hands with H and with nobody else.

  Person G does 7 handshakes and must therefore shake hands with everyone except A. Person B has already used up 2 handshakes (with H and G) and cannot shake anyone else's hand.

  Person F does 6 handshakes and must therefore shake hands with everyone except A and B. Person C has already used up 3 handshakes (with H, G, and F) and cannot shake anyone else's hand.

  Person E does 5 handshakes and must therefore shake hands with everyone except A, B, and C. Person D has already used up 4 handshakes (with H, G, F, and E) and cannot shake anyone else's hand.

  We see that Quintijn shakes hands with exactly four people: H, G, F, and E.

We conclude that in both cases, Quintijn shakes hands with four people.

A6. E) 34 The condition that n is divisible by 4, that $n+1$ is divisible by 5, and that $n+2$ is divisible by 6 is equivalent to demanding that $n-4$ is divisible by 4, 5, and 6. Hence, we are looking for numbers n for which $n-4$ is divisible by the least common multiple of 4, 5, and 6, which is 60. Thus, we find the numbers

$$4,\ 64,\ 124,\ 184,\ \ldots,\ 1984,$$

where $1984 = 4 + 33 \times 60$. In total we obtain 34 numbers.

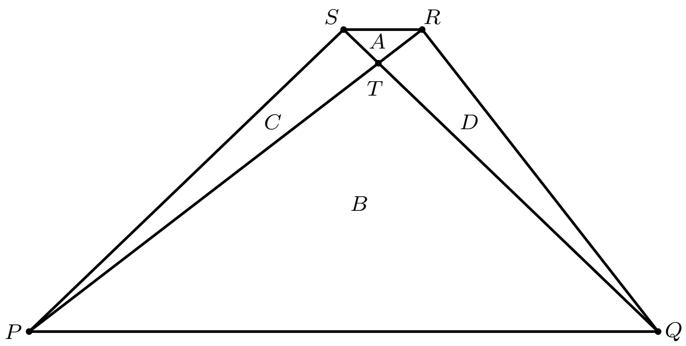
A7. E) only the snail No matter which type of jumps the frog makes, the x-coordinate of the frog remains even. Indeed, a jump of type 1 doesn't change the x-coordinate at all, while jumps of type 2 and 3 change the x-coordinate by -2 and 4, respectively. Therefore, the frog cannot reach the worm.

Now consider the value $x - y$. For every point that the frog can reach, this number must be a multiple of 5. Indeed, at the start we have $x - y = 0$ and upon a jump of type 1, 2, or 3, the value of $x - y$ changes by $0 - (-5) = 5$, by $-2 - 3 = -5$, and by $4 - 9 = -5$, respectively. This shows that the frog cannot reach the beetle.

The frog can reach the snail. To see this, we combine jumps of type 1 and 3. Together, they move the frog from position (x, y) to $(x + 4, y + 4)$. By doing this combination a total of 505 times, the frog lands on position $(2020, 2020)$. Finally, one jump of type 2 will bring the frog to $(2018, 2023)$, the position of the snail.

A8. E) $A : C = D : B$ We denote the vertices of the trapezium by P, Q, R, and S and denote the intersection of the two diagonals by T, see the figure. With respect to diagonal PR, triangles TRS and TPS have equal heights. This implies that the ratio of their areas, $A : C$, is the same as the ratio $|TR| : |TP|$. Similarly, D and B have the same ratio $|TR| : |TP|$. It follows that $A : C = D : B$. This is true for any trapezium that Harold may draw, hence option E) is correct.

To show that the other options are incorrect, consider the trapezium in the figure. Clearly, B is much larger than the other three areas. Therefore, options A), B), and C) are ruled out. Triangles PRS and QRS have the same base and height and therefore have the same area. It follows that also C and D are equal (this is true for any trapezium that Harold may draw). We see that $D : C$ is equal to 1, while $A : B$ is much smaller than 1. This rules out option D).



B1. 33 We know that three years ago, Rosa's mother was exactly five times as old as Rosa was at the time. From the information given, we also deduce that at that time, Rosa's grandmother was twice as old as Rosa's mother. We conclude that Rosa's grandmother was ten times as old as Rosa was at that time.

Let x be Rosa's current age. Three years ago, her grandmother's age was $10(x - 3)$. We are given that she is now 7 times as old as Rosa. This means that three years ago, she was $7x - 3$ years old. We thus find that $10(x - 3) = 7x - 3$. Simplifying the expression gives $3x = 27$ and hence $x = 9$. We see that three years ago, Rosa was $9 - 3 = 6$ years old. At that time, Rosa's mother was $5 \times 6 = 30$ years, which means that she is 33 years old now.

B2. 49998 Let x be the common starting number of Mike and Nanda, and let c be the digit that Mike places in front of his number. Nanda's number is therefore $10x + 4{,}000{,}008$ and Mike's number is $x + c \cdot 100{,}000$. Since Nanda's number is six times as large as Mike's number, we obtain $10x + 4{,}000{,}008 = 6x + 6c \cdot 100{,}000$. Simplifying this gives $4x = c \cdot 600{,}000 - 4{,}000{,}008$, hence $x = 150{,}000 \cdot c - 1{,}000{,}002$.

For $c \leqslant 6$ we obtain $x \leqslant 150{,}000 \cdot 6 - 1{,}000{,}002 = 900{,}000 - 1{,}000{,}002 < 0$. Since x is positive, we must have $c > 6$.

For $c \geqslant 8$ we obtain $x \geqslant 150{,}000 \cdot 8 - 1{,}000{,}002 = 1{,}200{,}000 - 1{,}000{,}002 = 199{,}998$. This number is too large since x must be a five-digit number. We must therefore have $c < 8$.

The only remaining option is therefore $c = 7$. For x we then find the value $x = 150{,}000 \cdot 7 - 1{,}000{,}002 = 49{,}998$.

B3. 15 The length of the diagonal of the square is $\sqrt{2}$ times the length of its sides. Hence, the ratio of the diameters of the two circles is $1 : \sqrt{2}$. The ratio of the areas of the two circles is therefore $1 : 2$ as the area scales with the square of the diameter.

Since the large circle has twice the area of the small circle, the four dark pieces and the four light pieces have a combined area that is equal to that of the small circle.

The small circle and the four dark pieces together form the square and hence have a combined area of 60. Combining this with the previous remark we find that the combined area of four light pieces and eight dark pieces also equals 60.

Dividing by four, we see that the combined area of one light piece and two dark pieces equals $\frac{60}{4} = 15$.

B4. 19 In any corner of a dubious die, three different numbers meet. Adding them gives a number that is at least $1 + 2 + 3 = 6$. Considering two opposite corners, we count the numbers on each face exactly once. Therefore, the six numbers on the faces sum to at least $2 \times 6 = 12$. Similarly, we see that the six numbers on the faces sum to at most $4 + 5 + 6 + 4 + 5 + 6 = 30$.

We will show that each of the numbers from 12 to 30 can indeed be obtained by summing the six numbers on the faces of some dubious die. Consider the three pairs of opposite faces. For each of the faces in the first pair we choose 3 or 4 as number. For each of the faces of the second pair we choose 2 or 5, and for each of the faces of the third pair we choose 1 or 6. In each corner we will then have exactly one number from each of the three pairs $\{3, 4\}$, $\{2, 5\}$, and $\{1, 6\}$. The result is therefore always a dubious die.

Adding the two numbers on the first pair of faces we can obtain the possible outcomes $6, 7, 8$. For the second pair the outcomes are $4, 7, 10$, and for the third pair they are $2, 7, 12$. Adding the four numbers from the first two pairs, the possible outcomes are therefore $4 + 6, 4 + 7, 4 + 8,\ 7 + 6, 7 + 7, 7 + 8,\ 10 + 6, 10 + 7, 10 + 8$, that is, the numbers 10 to 18. Hence, for the sum of all six numbers the possible outcomes are the numbers 12 to 20, 17 to 25, and 22 to 30. Each of the numbers from 12 to 30 can therefore occur as the sum of the numbers on a dubious die, and we obtain a total of 19 possible sums.