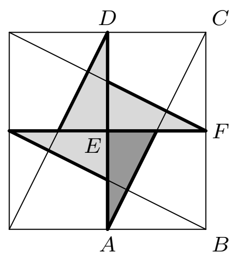
First round
Dutch Mathematical Olympiad

18 January – 28 January 2016

Solutions

A1. C) 2 Frank's two numbers are either both odd or both even, otherwise their sum is not even. The sum of Kees's two numbers is $41 - 26 = 15$. Hence of his numbers one must be even and one must be odd, as otherwise their sum is not odd. Pieter's numbers add up to $58 - 41 = 17$, hence of his numbers exactly one is even as well. It follows that the minimum number of even numbers is 2. An example would be that Frank has the numbers 11 and 15, Kees has the numbers 7 and 8, and Pieter has the numbers 4 and 13.

A2. D) 36 First we consider the rectangle $ABCD$ and the line segment EF that connects the midpoints of BC and AD. The diagonal AC intersects this line segment EF exactly in the middle. The star-shaped figure can be divided into four triangles, as done in the figure on the right. As we have just seen, the dark coloured triangle on the bottom right corner has height 6 and base length 3. Its area is therefore $\frac{1}{2} \cdot 3 \cdot 6 = 9$. This is also true for the other three triangles and hence the total area of the star-shaped figure is 36.

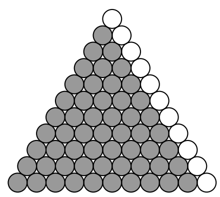


A3. B) 5 First consider the numbers starting with the digit 1. A number is always divisible by 1, hence this requirement is met. Of the numbers 12, 13, 14, 15, 16, 17, 18, and 19, the only two numbers that are divisible by their second digit are 12 and 15.

Now consider the numbers starting with the digit 2. To be divisible by 2, the last digit of the number must be *even*. Hence, the only candidates are 24, 26, and 28. We see that only 24 is fully divisible. For the numbers starting with the digit 3 we have two candidates: 36 and 39 (the others are not divisible by 3). Of these only 36 is fully divisible. For the numbers starting with the digit 4, the only candidate is 48 and it is indeed fully divisible. For numbers whose starting digit is 5 or greater, there is no second digit such that the number is divisible by the first digit. For example, for the starting digit 5 the only possibility would be 55, but this number is not allowed because the two digits have to be distinct.

We conclude that there are precisely 5 fully divisible two-digit numbers: 12, 15, 24, 36, and 48.

A4. E) 165 First, we count the horizontal eights (∞). The left circle of such an eight must be in one of the grey circles, see the figure. On the other hand, we can make a horizontal eight by choosing any grey circle and take it together with its right hand neighbour. Hence, the number of horizontal eights equals the number of grey circles and this is $66 - 11 = 55$. Because of symmetry, there are also 55 of each of the two types of diagonal eights (8 and 8). Hence, there are a total of $3 \cdot 55 = 165$ eights.



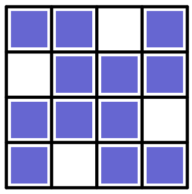
A5. B) 2 If three or more of the numbers are a multiple of three, then there are two neighbouring multiples of three and these add up to a multiple of three. This is not allowed, hence we conclude that there are at most two multiples of three along the circle.

Suppose that at most one of the numbers is a multiple of three. Then there are four successive numbers along the circle that are not a multiple of three. These numbers cannot all have the same residue when dividing by 3, as otherwise the sum of three successive numbers would be divisible by 3. For example, if the numbers are 1, 4, 7, and 10, then $1 + 4 + 7 = 12$ is divisible by 3. Of the four numbers, there is at least one which has residue 1 and at least one which has residue 2 when dividing by 3. In particular, we can find two neighbouring numbers of which one has residue 1 and the other has residue 2 when dividing by 3. But then their sum is divisible by 3, which is not allowed. We conclude that at least two of the numbers are a multiple of three.

By the previous reasoning, the only possibility is that exactly two of the numbers are a multiple of three. Taking the five numbers $3, 4, 7, 6$, and 1 in clockwise order around the circle shows that there is indeed a solution with exactly two numbers that are divisible by three. Indeed: $3 + 4$, $4 + 7$, $7 + 6$, $6 + 1$, and $1 + 3$ are not divible by 3, and neither are $3 + 4 + 7$, $4 + 7 + 6$, $7 + 6 + 1$, $6 + 1 + 3$, and $1 + 3 + 4$.

A6. E) 3630 Consider a wire-frame model of a $10\times10\times10$-cube. First, we count the wires going from the front to the back. These form 11 rows of 11 wires and hence there are $11\times11 = 121$ wires going from the front to the back. There are also 121 wires going from left to right and 121 going from the top to the bottom. Each of these wires has a length of 10 dm. Hence, the total length of the wires is 3630 dm.

A7. D) 12 A colouring with 12 blue squares in which each blue square is adjacent to exactly one white square can be found in the figure. Such a colouring having more than 12 blue squares does not exist. In a colouring having 13 or more blue squares there are at most 3 white squares. These white squares together have at most $3\times4 = 12$ neighbouring squares, hence at least one of the blue squares does not have a white neighbouring square.



A8. D) $b + c - a < 3$ We first determine all possible solutions (a, b, c) of the inequality $a+2b+3c < 12$, where a, b and c are distinct positive integers. From $3c < 12 - a - 2b \leqslant 9$ it follows that $c < 3$, hence $c = 2$ or $c = 1$.

- **Case** $c = 2$ We have $a + 2b < 6$, hence $b < 3$. Because b and c cannot be equal, b must equal 1. We now find exactly one solution: $(a, b, c) = (3, 1, 2)$.
- **Case** $c = 1$ We have $2b < 9 - a \leqslant 8$, hence $b < 4$. If $b = 3$, then we have $a < 3$ and a cannot be equal to 1. Hence we find $(a, b, c) = (2, 3, 1)$ as the only solution in this case. If $b = 2$, then we have $a < 5$, hence $a = 3$ or $a = 4$, because a cannot equal 1 or 2. We find the two solutions $(3, 2, 1)$ and $(4, 2, 1)$.

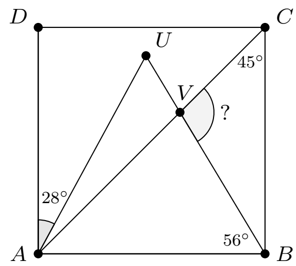
The only solutions (a, b, c) are $(3, 1, 2)$, $(2, 3, 1)$, $(3, 2, 1)$, and $(4, 2, 1)$. Because $(4, 2, 1)$ does not satisfy the inequalities of A) and B), these answers are wrong. Because $(3, 1, 2)$ does not satisfy the inequalities of C) and E), these are wrong as well. However, the four solutions do satistfy the inequality of D), hence that answer is the correct one.

B1. 70707 Suppose that 707070 is divisible by d. Then 707070 is also divisible by $\frac{707070}{d}$. The seven smallest numbers dividing 707070 are $1, 2, 3, 5, 6, 7, 10$. Hence, the seventh number in the list is $\frac{707070}{10} = 70707$.

B2. 'AAOOOOAA' (or 'OOAAAAOO') A word in the AO-language is an alternation of blocks of A's and blocks of O's. For example: the word 'AA O AAA OOO AA O' consists of three A-blocks and three O-blocks. The letter combinations 'AO' precisely correspond to switching from an A-block to an O-block. The letter combinations 'OA' correspond to switching from an O-block to an A-block. The word in the example above, therefore, has three combinations 'AO' and two combinations 'OA'. A word that starts and end with the same letter switches from A-block to O-block equally often as it switches the other way around. Therefore, it has an equal number of 'AO' and 'OA' combinations. A word starting with an A and ending in O has one extra combination 'AO', whereas a word starting with O and ending with A has one extra combination 'OA'.

It follows that a word in the AO-language is special if and only if the first and last letter are equal. Hence, we are looking for a word with the property that the first and last letter are equal. Moreover, this property must be retained when you remove the first or the last letter. This means that the first two and the last two letters of the word are all equal. The only two words consisting of four A's and four O's having that property are AAOOOOAA and OOAAAAOO.

B3. 101° We first consider triangle ABU. The angle at A has size $90° - 28° = 62°$. Because tiangle ABU is isosceles, also the angle at U has size 62°. Because the sum of the angles of a triangle equals 180°, the angle at B has size $180° - 62° - 62° = 56°$.



Now we consider triangle ABC. Triangle ABC is isosceles. The two equal angles at A and C plus the right angle at B sum up to 180°. Hence, the angle at C has size $\frac{180°-90°}{2} = 45°$.

To conclude, we consider BCV. The angle at C has size 45°. The angle at B has size $90° - 56° = 34°$. Hence, the angle at V has size $180° - 45° - 34° = 101°$.

B4. Denise First, we consider the statement of Helga. Alex and Eva have the same eye colour and the same hair colour. The same holds for Chris and Denise. Boris, Felix, and Gaby, on the other hand, have a unique combination of eye and hair colour. From the statement of Helga we can deduce that Boris, Felix, and Gaby are not the thief, because otherwise Helga would have deduced this from their eye and hair colour. Alex, Chris, Denise, and Eva, however, are still suspects, because they do not have a unique combination of eye and hair colour.

Now, we consider the statement of Ingrid. Alex and Felix have the same hair colour and the same gender, just like Boris and Chris, and also Denise and Gaby. Only Eva has a unique combination of hair colour and gender. From the statement of Ingrid, we can deduce that Eva is not the thief, because otherwise Ingrid would have known that Eva is the thief. The other six are not excluded by Ingrid's statement to be the thief.

Based on the statements of Helga and Ingrid, Julius concludes that Boris, Eva, Felix, and Gaby are *not* the thief. Without extra information, each of Alex, Chris, and Denise could possibly be the thief. Julius has some extra data, the gender of the thief, by which he can now determine the thief. If the thief were male, then he would have only been able to conclude that either Alex or Chris is the thief. Hence, the thief must be female, namely Denise.