Second round
Dutch Mathematical Olympiad

Friday 15 March 2013

Solutions

B-problems

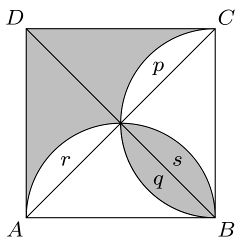
B1. 13 A solution with thirteen students is possible. If five students scored 100 points and the remaining eights students scored 61 points, the average score equals $\frac{5\cdot 100+8\cdot 61}{13} = \frac{988}{13} = 76$ points, as required.

It is not possible that the number of students taking the test was twelve or less. Indeed, suppose that $n \leqslant 12$ students took the test. Five of them scored 100 points and the remaining $n-5$ students scored at least 60 points. Their total score would be at least $500+(n-5)\cdot 60 = 60n+200$. Their average score would then be at least

$$\frac{60n+200}{n} = 60 + \frac{200}{n} \geqslant 60 + \frac{200}{12} = 76\tfrac{2}{3},$$

since $n \leqslant 12$. This, however, contradicts the fact that their average score was 76.

B2. 8 Note that both circles go through the middle of the square. Therefore, the four circle segments indicated by p, q, r, and s all belong to one fourth of a circle with radius 2, hence the four segments have equal areas. Therefore, the combined area of the grey shapes equals the area of triangle ACD which is $\frac{1}{2}\cdot 4\cdot 4 = 8$.



B3. 5 o'clock In 12 hours, the amounts by which the hour hands of the first and the second clock are ahead increase by $\frac{1}{100}$ and $\frac{5}{100}$ of a turn respectively. Therefore, in 12 hours, the second clock increases its lead compared to the first clock by $\frac{5-1}{100} = \frac{1}{25}$ of a full turn. Hence after $25\cdot 12$ hours, the hour hand of the second clock has made exactly one extra full turn compared to that of the first clock. This is the first time the two clocks again display the same time.

During those $25\cdot 12$ hours, the hour hand of the first clock has made exactly $\frac{101}{100}\cdot 25 = 25\frac{1}{4}$ full turns. Both clocks will then display a time of $2+3=5$ o'clock.

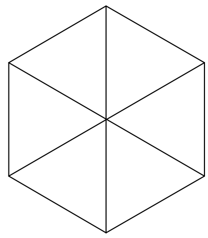
B4. $\frac{1}{4}$ The product of the eight numbers in the second and fourth row equals the product of the eights numbers in the first and second column. Writing this out, we get:

$\frac{1}{2}$	32	8	1
4	2	8	2
4	1	8	4
16	2	$\frac{1}{4}$	16

$$C\cdot 2\cdot 8\cdot 2\cdot F\cdot G\cdot H\cdot 16 = \tfrac{1}{2}\cdot C\cdot 4\cdot F\cdot 32\cdot 2\cdot 1\cdot G.$$

Since C, F, and G are nonzero, we may divide both sides of the equation by C, F, and G. The resulting equation is $512\cdot H = 128$, which implies that $H = \frac{1}{4}$. The figure shows one solution.

B5. 19 By dividing the regular hexagon into six equilateral triangles, we deduce that the length of the long diagonal AC of the hexagon equals twice the side length of the hexagon. We will compute three times the side length, namely $|AB| + |BC| + |CD|$. Observe that AB is a side of a parallelogram with the parallel side having length $11 + 16 = 27$. Thus we have $|AB| = 27$.

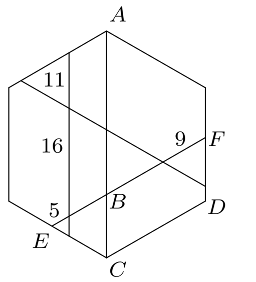


As triangle BCE is equilateral, we have $|BC| = |EB|$.

As $BCDF$ ia a parallelogram, we have $|CD| = |BF|$.

From the figure we see that $|EB| + |BF| = |EF| = 5 + 16 + 9 = 30$.

If we combine these facts, we find that three times the side length of the regular hexagon equals



$$|AB| + |BC| + |CD| = 27 + |EB| + |BF| = 27 + 30 = 57.$$

The side length therefore equals $\frac{57}{3} = 19$.

C-problems

C1. (a) The number 4132 starts with a '4' and is above average because $2 \cdot 3 \geqslant 4+1$ and $2 \cdot 2 \geqslant 1+3$.

(b) Suppose that $a4bc$ is a 4-digit number that is above average, where a, b, and c are the digits '1', '2', and '3' (possibly in a different order). Then $2 \cdot b \geqslant a + 4 \geqslant 5$. So $b \geqslant 3$. Similarly, we find that $2 \cdot c \geqslant 4 + b \geqslant 7$, hence $c \geqslant 4$. However, this is impossible because c was at most 3.

(c) The numbers 1243756, 1234576, and 1234567 are above average and have digit '7' in the fifth, sixth, and seventh position, respectively.

Digit '7' cannot be in the first position. Indeed, suppose that $7abcdef$ would be above average. Then $2 \cdot b \geqslant 7 + a \geqslant 8$, hence $b \geqslant 4$. Then we must have $2 \cdot c \geqslant a + b \geqslant 5$, hence $c \geqslant 3$. Now we find (in turn) that also $d, e, f \geqslant 4$. It follows that both digit '1' and digit '2' must be in the position of a, which is impossible.

Digit '7' cannot be in the second or third position. Indeed, otherwise the digit following '7' must be at least 4, which implies that also the digits following it must be at least 4. Digits '1', '2', and '3' must therefore all be in the first two positions, which is impossible.

Finally, digit '7' cannot be in the fourth position. Digit '1' cannot be in the third position since $2 \cdot 1 < 2 + 3$. Because the digit in the third position must be at least 2, the digit in the fifth position must be at least 5. The next digit must therefore be at least 6, as must be the digit following it. The digits '1' to '4' must therefore all be in the first three positions, which is impossible.

C2. (a) Since $x \geqslant 5$ is odd, we can write $x = 2n + 1$ for an integer $n \geqslant 2$. Now

$$(x, y, z) = (2n + 1, n, n + 2) \quad \text{and} \quad (x, y, z) = (2n + 1, n + 1, n - 1)$$

are two different good triples. In both cases it is clear that y and z are indeed positive integers and that $y \geqslant 2$. Substitution into the equation shows that they are indeed solutions:

$$(2n + 1)^2 - 3n^2 = n^2 + 4n + 1 = (n + 2)^2 - 3 \quad \text{and}$$
$$(2n + 1)^2 - 3(n + 1)^2 = n^2 - 2n - 2 = (n - 1)^2 - 3.$$

Thic concludes the solution.

Remark. *One way to arrive at the idea of considering these triples is the following. First substitute $x = 5$. It is then easy to see that z can be no more than 5. The case $z = 5$ is* *not possible, because then $y = 1$, which is not allowed. Hence z is at most 4. For $z = 1, \ldots, 4$ compute the corresponding value of y, if it exists. This way, you will find two good triples with $x = 5$. Repeating this for $x = 7$ and $x = 9$, you will find good triples as well. The triples found show a clear pattern: when x increases by 2, y and z both increase by 1. This holds for both series of triples. Using this, you can guess a general expression for y and z when $x = 2n + 1$. Checking that the found triples are good by substitution in the equation will then suffice for a complete solution.*

(b) An example is triple $(16, 9, 4)$. This triple is good because $16^2 - 3 \cdot 9^2 = 13 = 4^2 - 3$.

Remark. *Stating a suitable triple and showing that it is a good triple suffices for a complete solution. To find such a triple, one possibility is to take the following approach. Rewrite the equation as $x^2 - z^2 = 3y^3 - 3$. Both sides of the equation can be factored, which gives you : $(x - z)(x + z) = 3(y - 1)(y + 1)$. This will help in finding triples by substituting different values for y. For example, you can try $y = 4$. Then the right-hand side becomes $3 \cdot 3 \cdot 5$, hence the left-hand side becomes $5 \cdot 9$, $3 \cdot 15$, or $1 \cdot 45$. The value of x will always be the average of the two factors, so $x = 7$, $x = 9$, and $x = 23$ in these three cases. There are no even values for x when $y = 4$. If you try further values of y, you will find even values of x for $y = 7$ and $y = 9$.*