Round 2
Dutch Mathematical Olympiad

Friday 25 March 2011

Solutions

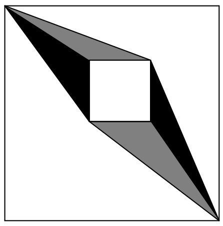
B-problems

B1. $\frac{12}{19}$ Let w be the number of women present, and let m be the number of men present. The problem tells us that $\frac{2}{3}w = \frac{3}{5}m$ and hence that $w = \frac{9}{10}m$. The number of people dancing, is exactly twice the number of men dancing, namely $\frac{6}{5}m$.

The number of people present is of course $m + w = m + \frac{9}{10}m = \frac{19}{10}m$. So it follows that the part of those present that is dancing, is equal to

$$\frac{\frac{6}{5}m}{\frac{19}{10}m} = \frac{6}{5} \cdot \frac{10}{19} = \frac{12}{19}.$$

B2. 10 We have split the black part into four triangles, and have coloured two of them gray. The two gray triangles both have base 2, and their combined height is $7 - 2 = 5$, namely the height of the larger square minus the height of the smaller square. Hence the area of the two grey triangles together is equal to $\frac{1}{2} \cdot 2 \cdot 5 = 5$. The same holds for the two black triangles. It follows that the combined area is $5 + 5 = 10$.



B3. 7 There are 23 students in total. From what's given, it follows that:

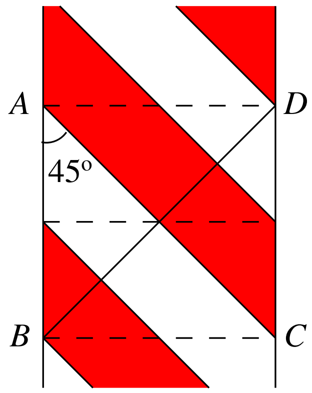
$$\begin{aligned}
16 + 11 + 10 &= (\text{girls with French} + \text{boys with German}) + \text{everyone with French} + \text{all girls} \\
&= (\text{girls with French} + \text{boys with German}) \\
&\quad + (\text{girls with French} + \text{boys with French}) \\
&\quad + (\text{girls with French} + \text{girls with German}) \\
&= 3 \times \text{girls with French} + \text{boys with German} \\
&\quad + \text{boys with French} + \text{girls with German} \\
&= 2 \times \text{girls with French} + 23.
\end{aligned}$$

So the total number of girls that have chosen French is equal to $\frac{16+11+10-23}{2} = \frac{14}{2} = 7$.

B4. 198 In the first step, we remove the cards numbered by $1^2, 2^2, 3^2, \ldots, 100^2$. Then 9900 cards remain. Since $99^2 \leqslant 9900 < 100^2$, we remove $1^2, 2^2, \ldots, 99^2$ in the second step. After that, $9900 - 99 = 9801 = 99^2$ cards are left, which is a square.

In general, if we start with n^2 cards, with $n \geqslant 2$, we remove n cards in the first step, after which $n^2 - n$ cards remain. Since $(n-1)^2 = n^2 - 2n + 1 \leqslant n^2 - n < n^2$, we remove $n - 1$ cards in the second step. Then exactly $(n^2 - n) - (n - 1) = (n-1)^2$ are left. So in two steps we can reduce the number of cards from n^2 to $(n-1)^2$. It follows that we need $2 \cdot 99 = 198$ steps to remove all but one of the cards when we start with 100^2 cards.

B5. $\pi\sqrt{2}$ cm Imagine the pole as a paper cylinder. Cut it open along its length, then unroll it, to get a rectangular strip of paper. So points A and D correspond to the same point on the cylinder, just like points B and C. The width of the strip is equal to the perimeter of the cylinder, so $|AD| = |BC| = 2\pi \cdot 2$ cm $= 4\pi$ cm.



Note that the red ribbon forms a 45° angle with the cutting line, $ABCD$ is a square. The length of the diagonal BD is equal to $\sqrt{2} \cdot 4\pi$ cm and also equal to four times the width of the red ribbon, since the white and red stripes have the same width. It follows that the red ribbon has width $\pi\sqrt{2}$ cm.

C-problems

C1. Since a, b and c are three successive positive odd integers, we can write: $a = 2n-1$, $b = 2n+1$ and $c = 2n+3$, with n a positive integer. A calculation then gives:

$$\begin{aligned} a^2+b^2+c^2 &= (2n-1)^2+(2n+1)^2+(2n+3)^2 \\ &= (4n^2-4n+1)+(4n^2+4n+1)+(4n^2+12n+9) \\ &= 12n^2+12n+11. \end{aligned}$$

This needs to be equal to an integer that consists of four digits p. Hence the integer $12n^2+12n$ consists of four digits, of which the first two are equal to p, and the last two are equal to $p-1$. Since $12n^2+12n$ is divisible by 2, $p-1$ has to be even. So we have the following possibilities for $12n^2+12n$: 1100, 3322, 5544, 7766 and 9988. This integer must be divisible by 3, so the only integer remaining is 5544, so $n^2+n = \frac{5544}{12} = 462$. We can rewrite this as $n^2+n-462 = 0$. Factorizing this quadratic equation then gives: $(n-21)(n+22) = 0$. Since n is a positive integer, the only solution is $n = 21$. So the only triple satisfying the given properties is $(a,b,c) = (41,43,45)$.

C2. Note that the possible scores are multiples of 5. The lowest score a student can get is 0, and the highest score is $16 \cdot 10 = 160$. Now suppose that there are no two students with the same score. Then the combined score of the students is at most $160+155+150+\cdots+15 = \frac{1}{2}\cdot 175\cdot 30 = 2625$. We'll derive a contradiction from this.

Let A be the combined number of correct answers that were given within one minute, B be the combined number of correct answers that were not given within a minute, and C be the combined number of incorrect answers. The students answered $16\cdot 30 = 480$ questions together, so $A+B+C = 480$. More than half of the questions was answered correctly within one minute, so $A > 240$. Also note that $B = C$, so $B = C = \frac{480-A}{2}$. We now can express the combined score in A. This is equal to:

$$10\cdot A + 5\cdot B + 0\cdot C = 10\cdot A + 5\cdot \frac{480-A}{2} = \tfrac{15}{2}A + 1200.$$

Since $A > 240$, the combined scores of the students is greater than $\frac{15}{2}\cdot 240 + 1200 = 3000$. But from the assumption that no two students have the same score, we deduced that the combined score was at most 2625. This is a contradiction. We deduce that this assumption was wrong, so that there are two students with the same score.