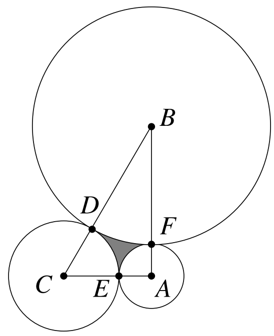
Final round
Dutch Mathematical Olympiad

Friday 17 September 2010, Technical University Eindhoven

Solutions

1. We recognize triangle ABC to be half an equilateral triangle. This implies that $|BC| = 2|AC| = 12$. The Pythagorean theorem yields: $|AB| = \sqrt{|BC|^2 - |AC|^2} = \sqrt{108} = 6\sqrt{3}$.

Denote the pairwise tangent points of the three circles by D, E and F (see figure) and the radii of the three circles by r_A, r_B and r_C. The strategy will be to determine the area of the three circular sectors and subtract them from the area of triangle ABC.



We see that $2r_A = (r_A+r_C)+(r_A+r_B)-(r_B+r_C) = |AC|+|AB|-|BC| = 6\sqrt{3}-6$, so $r_A = 3\sqrt{3}-3$. It follows that $r_B = 6\sqrt{3}-r_A = 3\sqrt{3}+3$ and $r_C = 6 - r_A = 9 - 3\sqrt{3}$.

The area of a circle of radius r equals πr^2. Therefore, the area of circular sector AFE equals $\frac{90}{360}\cdot \pi r_A^2$, or $\frac{1}{4}\pi(36-18\sqrt{3}) = 9\pi - \frac{9}{2}\sqrt{3}\pi$. For the area of circular sectors BDF and CED we find, respectively, $\frac{30}{360}\pi r_B^2 = 3\pi + \frac{3}{2}\sqrt{3}\pi$ and $\frac{60}{360}\pi r_C^2 = 18\pi - 9\sqrt{3}\pi$.

Since ABC has an area of $\frac{1}{2}\cdot|AB|\cdot|AC| = 18\sqrt{3}$, we obtain a value of $18\sqrt{3} - (9\pi - \frac{9}{2}\sqrt{3}\pi) - (3\pi + \frac{3}{2}\sqrt{3}\pi) - (18\pi - 9\sqrt{3}\pi) = 18\sqrt{3} - 30\pi + 12\sqrt{3}\pi$ for the area of the gray region. □

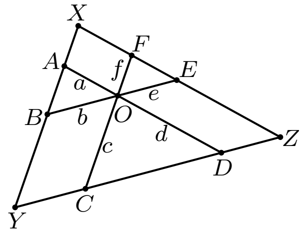
2. (a) Suppose that $k = m + (m+1) + \cdots + (n-1) + n$ is a polite number. The sum formula for arithmetic sequences gives $k = \frac{1}{2}(m+n)(n-m+1)$. As m and n are different positive numbers, $m+n \geqslant 3$ and $(n-m)+1 \geqslant 2$ must hold.

Since $(m+n)+(n-m+1) = 2n+1$ is odd, one of the numbers $m+n$ and $n-m+1$ is odd. Hence $2k = (m+n)(n-m+1)$ has an odd divisor (greater than 1) and is therefore not a power of two. This implies that k is not a power of two either.

We conclude that no number can be both polite and a power of two.

(b) Suppose that k is a positive integer, not a power of two. We will show k to be a polite number. Collecting all factors 2, we can write $k = c\cdot 2^d$, where c is odd and $d \geqslant 0$ is a nonnegative integer. The assumption that k is not a power of two, means that $c > 1$. We need to find $n > m$ such that $m + \cdots + n = \frac{1}{2}(m+n)(n-m+1) = c \cdot 2^d$, or $(m+n)\cdot(n-m+1) = c\cdot 2^{d+1}$. We can achieve this by choosing m and n in such a way that $m+n = c$ and $n-m+1 = 2^{d+1}$, or conversely: $m+n = 2^{d+1}$ and $n-m+1 = c$. To ensure that m will be positive, we consider two cases.
For $c \geqslant 2^{d+1}$ we solve: $m+n = c$, $n-m+1 = 2^{d+1}$. This gives $m = (c-2^{d+1}+1)/2$ and $n = (c+2^{d+1}-1)/2$. Obviously, $n > m$ (since $2^{d+1} \geqslant 2$). Both m and n are integers (the numerators are even since c is odd) and positive by the assumption $c \geqslant 2^{d+1}$.
For $c < 2^{d+1}$ we solve: $m+n = 2^{d+1}$, $n-m+1 = c$. This gives $m = (2^{d+1}-c+1)/2$ and $n = (2^{d+1}+c-1)/2$. Clearly, $n > m$ holds (since $c > 1$) and both m and n are positive integers. □

3. Since AO and XZ are parallel, $\angle OAB = \angle ZXY$ are corresponding angles. Similarly, since BO and YZ are parallel, $\angle ABO = \angle XYZ$ holds. We deduce that $\triangle OAB \sim \triangle ZXY$ (equal angles). Hence there is a scaling factor u such that $a = u|XZ|$ and $b = u|YZ|$. Using similar arguments we find that $\triangle OCD \sim \triangle XYZ$ and $\triangle OEF \sim \triangle YZX$. So there are scaling factors v and w such that $c = v|XY|$, $d = v|XZ|$, $e = w|YZ|$ and $f = w|XY|$.



We now see that $a\cdot c\cdot e = uvw\cdot|XY|\cdot|YZ|\cdot|ZX| = b\cdot d\cdot f$. This implies that $a\cdot b\cdot c\cdot d\cdot e\cdot f = (a\cdot c\cdot e)^2$, which is a perfect square since a, c and e are integers. $\square$

4. (a) Suppose that (x, y) is such a pair and consider the integers $a = x + 3y$ and $b = 3x + y$. From $0 < x, y < 1$ it follows that $0 < a, b < 4$, or: $1 \leqslant a, b \leqslant 3$.

Conversely, let a and b be integers such that $1 \leqslant a, b \leqslant 3$. There is a unique pair of numbers (x, y) that satisfies $a = x + 3y$ and $b = 3x + y$. Indeed, combining the two equations, we get $3b - a = 3(3x + y) - (x + 3y) = 8x$ and $3a - b = 8y$. In other words $x = (3b - a)/8$ and $y = (3a - b)/8$ (and these x and y do satisfy the two equations). If we substitute $1, 2, 3$ for a and b, we obtain the folowing nine pairs (x, y):

$$(\tfrac{2}{8}, \tfrac{2}{8}), (\tfrac{5}{8}, \tfrac{1}{8}), (\tfrac{8}{8}, \tfrac{0}{8}), \quad (\tfrac{1}{8}, \tfrac{5}{8}), (\tfrac{4}{8}, \tfrac{4}{8}), (\tfrac{7}{8}, \tfrac{3}{8}), \quad (\tfrac{0}{8}, \tfrac{8}{8}), (\tfrac{3}{8}, \tfrac{7}{8}), (\tfrac{6}{8}, \tfrac{6}{8}).$$

The condition $0 < x, y < 1$ rules out the two candidates $(x, y) = (\frac{8}{8}, \frac{0}{8})$ and $(x, y) = (\frac{0}{8}, \frac{8}{8})$. This leaves the 7 pairs we were asked to find.

(b) Suppose that $0 < x, y < 1$ holds and that $a = x + my$ and $b = mx + y$ are integers. Then $1 \leqslant a, b \leqslant m$ holds.

Given integers a and b with $1 \leqslant a, b \leqslant m$, there is a unique pair (x, y) for which $x + my = a$ and $mx + y = b$ hold. Indeed, combining the two equalities gives : $mb - a = (m^2 - 1)x$ and $ma - b = (m^2 - 1)y$, or: $x = (mb - a)/(m^2 - 1)$ and $y = (ma - b)/(m^2 - 1)$. These x and y indeed satisfy the two equations.

For given a and b, we determine whether the corresponding numbers x and y satisfy $0 < x, y < 1$. From $1 \leqslant a, b \leqslant m$ it follows that $x \geqslant (m \cdot 1 - m)/(m^2 - 1) = 0$ and $x \leqslant (m \cdot m - 1)/(m^2 - 1) = 1$. The cases $x = 0$ and $x = 1$ exactly correspond to $(a, b) = (m, 1)$ and $(a, b) = (1, m)$ respectively. Similarly, $0 < y < 1$ holds, unless $(a, b) = (1, m)$ or $(a, b) = (m, 1)$. Among the m^2 possible pairs (a, b), there are exactly two for which (x, y) is not a solution. In total there are $m^2 - 2$ solutions (x, y).

From $m^2 - 2 = 119$, we see that $m = 11$. $\square$

5. A strategy that guarantees a win for Amber is as follows. In Amber's turn, she splits every pile with an even number of coins (say $2k$) in two piles with an odd number of coins: 1 coin and $2k - 1$ coin respectively. The piles having an odd number of coins, she leaves untouched. So in her first turn, she created one pile of 1 coin and one of 2009 coins.

When Brian gets to make a move, all piles will have an odd number of coins. He is therefore forced to split an odd pile, creating a new pile with an even number of coins. This implies that Amber, in het next turn, can continue her strategy, since there will be at least one even pile.

With each turn, the number of piles increases, so after at most 2009 turns, the game is over. Since Brian always creates an even pile, the game cannot end during his turn. Therefore, it will be Amber who wins the game. $\square$