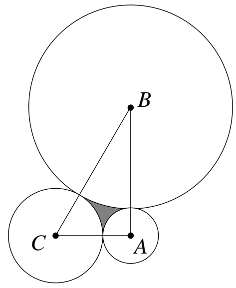
Final round
Dutch Mathematical Olympiad

Friday 17 September 2010
Technical University Eindhoven

- Available time: 3 hours.
- Each problem is worth 10 points. A description of your solution method and clear argumentation are just as important as the final answer.
- Calculators and formula sheets are not allowed. You can only bring a pen, ruler (set square), compass and your math skills.
- Use a separate sheet for each problem. Good luck!

1. Consider a triangle ABC such that $\angle A = 90°$, $\angle C = 60°$ and $|AC| = 6$. Three circles with centers A, B and C are pairwise tangent in points on the three sides of the triangle.
Determine the area of the region enclosed by the three circles (the grey area in the figure).

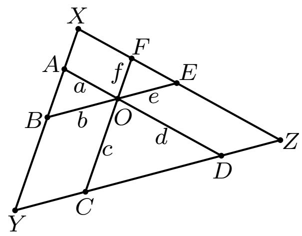


2. A number is called *polite* if it can be written as $m + (m+1) + \cdots + (n-1) + n$ for certain positive integers $m < n$. For example: 18 is polite, since $18 = 5 + 6 + 7$. A number is called a *power of two* if it can be written as 2^ℓ for some integer $\ell \geqslant 0$.

(a) Show that no number is both polite and a power of two.

(b) Show that every positive integer is polite or a power of two.

3. Consider a triangle XYZ and a point O in its interior. Three lines through O are drawn, parallel to the respective sides of the triangle. The intersections with the sides of the triangle determine six line segments from O to the sides of the triangle.
The lengths a, b, c, d, e and f of these segments are integers (see figure).
Prove that the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ is a perfect square.



4. (a) Determine all pairs (x, y) of (real) numbers with $0 < x < 1$ and $0 < y < 1$ for which $x + 3y$ and $3x + y$ are both integers. An example is $(x, y) = \left(\frac{3}{8}, \frac{7}{8}\right)$, since $x + 3y = \frac{3}{8} + \frac{21}{8} = \frac{24}{8} = 3$ and $3x + y = \frac{9}{8} + \frac{7}{8} = \frac{16}{8} = 2$.

(b) Determine the integer $m \geqslant 2$ for which there are exactly 119 pais (x, y) with $0 < x < 1$ and $0 < y < 1$ such that $x + my$ and $mx + y$ are integers.

Remark: if $u \neq v$, the pairs (u, v) and (v, u) are different.

5. Amber and Brian are playing a game using 2010 coins. Throughout the game, the coins are divided into a number of piles of at least 1 coin each. A move consists of choosing one or more piles and dividing each of them into two smaller piles. (So piles consisting of only 1 coin cannot be chosen.) Initially, there is only one pile containing all 2010 coins. Amber and Brian alternatingly take turns to make a move, starting with Amber. The winner is the one achieving the situation where all piles have only one coin.
Show that Amber can win the game, no matter which moves Brian makes.