First round
Dutch Mathematical Olympiad

21 January – 31 January 2013

Solutions

A1. B) 2 It is given that the light is red at 12:05 pm. The colours at times 12:05 pm to 12:12 pm are now fixed for a traffic light of period 1: they are alternately red and green. When the period is 2, there are two possibilities and for period 3 there are three possibilities:

period	12:05pm	12:06pm	12:07pm	12:08pm	12:09pm	12:10pm	12:11pm	12:12pm
1 min	red	green	red	green	red	green	red	green
2 min	red	red	green	green	red	red	green	green
2 min	red	green	green	**red**	**red**	green	green	red
3 min	red	red	red	**green**	**green**	green	red	red
3 min	red	red	green	**green**	**green**	red	red	red
3 min	red	green	green	green	red	red	red	green

In three of the six cases, the light is red at 12:12 pm, as required. This gives us two colour combinations for the light at 12:08 pm and at 12:09 pm: red–red and green-green.

A2. C) 120 Twice the length of a small rectangle equals three times its width. Therefore, the ratio between length and width equals $3:2$. As the perimeter is 20, the length must be 6 and the width must be 4. We see that the area of each small rectangle equals $6 \times 4 = 24$, hence the area of rectangle $ABCD$ is $5 \times 24 = 120$.

A3. B) b Comparing sums of pairs of numbers, we find:

$$e+a < c+d < a+b < b+c < d+e.$$

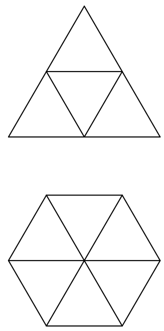
Every sum of four of the numbers can be obtained by adding two of the pairs. For example, $a+b+c+e$ is equal to $(e+a)+(b+c)$. Of all four-tuples, a,c,d,e has the smallest sum, because $a+c+d+e = (e+a)+(c+d)$ is the sum of the two smallest pairs. The remaining number b must be the largest among the five numbers. Indeed, the largest number is the one for which the remaining four numbers have the smallest sum.

A4. A) 3 The order in which the moves are made is irrelevant for the final result. By pressing the three bulbs in the top row, all bulbs will change from being *on* to being *off*. Indeed, the bulbs in the top row change their state three times, and the other bulbs change their state exactly once.

If is not possible to turn *off* all light bulbs by pressing two or fewer bulbs. Indeed some bulb will not be in the same row or column as the chosen bulbs, and hence remain *on*.

A5. C) $\frac{1}{15}$ Without changing the problem, we may assume that there are 20 boxes. Hence, a total of 5 boxes is empty. Out of the 5 boxes that are opened, one fifth turns out to be non-empty, exactly 1 box. As 4 of the opened boxes are empty, there is exactly 1 empty box among the remaining 15 unopened boxes.

A6. D) 3 : 2 We divide the hexagon into 6 equal equilateral triangles and divide the triangle into 4 equal equilateral triangles, see the figure. Since the hexagon and the triangle have the same perimeter, the sides of the triangle are twice as long as the hexagon's sides. Therefore, the triangles in both divisions have the same size. It follows that the ratio between the area of the hexagon and the area of the triangle equals 6 : 4, or: 3 : 2.



A7. C) 3125 We consider the last four digits of the powers of 5:

$$\begin{array}{rclrcr} 5^1 & = & 0005 & 5^5 & = & 3125 \\ 5^2 & = & 0025 & 5^6 & = & 1\,5625 \\ 5^3 & = & 0125 & 5^7 & = & 7\,8125 \\ 5^4 & = & 0625 & 5^8 & = & 39\,0625 \end{array}$$

The last four digits of a power of 5 are already determined by the last four digits of the previous power of 5. For example: the last four digits of 5×390625 and 5×0625 are both 3125. Because the last four digits of 5^4 and 5^8 are the same, the last four digits of powers of 5 will repeat every four steps. The last four digits of 5^{2013} will be the same as those of 5^{2009} and those of 5^{2005}, continuing all the way down to the last four digits of $5^5 = 3125$. We conclude that 3125 are the last four digits of 5^{2013}.

A8. B) 64 Since every student answered correctly a different number of questions, at least $0 + 1 + 2 + \cdots + 19 = 190$ correct answers were given in total. Because every question was answered correctly at most three times, there must be at least $\frac{190}{3} = 63\frac{1}{3}$ questions. That is, there were at least 64 questions.

A situation with 64 questions is indeed possible. Let the number of correct answers given by the twenty students be $0, 1, \ldots, 19$. We divide the students into three groups:

I consists of those students having 0, 1, 2, 3, 4, 17, 18, or 19 correct answers,

II consists of those students having 5, 6, 7, 14, 15, or 16 correct answers, and

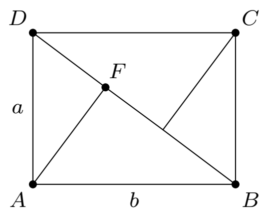
III consists of those students having 8, 9, 10, 11, 12, or 13 correct answers.

The total number of correct answers in each group is no more than 64. Within each group, we can therefore choose the correctly answered questions to be distinct. That way, no question is answered correctly more than three times.

B1. 244882 We only consider numbers composed of the even digits 2, 4, and 8. If such a number ends with digit 4 or 8, it is a multiple of 20 plus 4 or 8, and hence divisible by 4. If such a number ends with digit 2, it is a multiple of 20 plus 2, and hence not divisible by 4. So we have to choose 2 as the last digit. Sorting the remaining digits in increasing order, we obtain 244882.

B2. $\frac{3}{2}$ Triangles ADF, BDA en BAF are similar (AA). Hence

$$\frac{b}{a} = \frac{|BA|}{|DA|} = \frac{|AF|}{|DF|} = \frac{|BF|}{|AF|}.$$



Therefore

$$\left(\frac{b}{a}\right)^2 = \frac{|AF|}{|DF|} \cdot \frac{|BF|}{|AF|} = \frac{|BF|}{|DF|} = \frac{9}{4}.$$

We conclude that $\frac{b}{a} = \frac{3}{2}$.

B3. 30 minutes The time needed by Fred to bike from the middle stop to the last stop and run back to the middle stop, is equal to the time he needs to bike from the middle stop to the first stop plus the time he needs to run from the middle stop to the last stop. This is because the distance to both stops is the same.

It is given that this amount of time equals the time the bus needs to get to the first stop plus the time it needs to get to the last stop. This is exactly twice the time the bus needs to get to the middle stop: $2 \times 15 = 30$ minutes.

B4. 25 times The combination of digits "2013" occurs 13 times as part of the following numbers: 2013, 12013, 22013, and 20130 to 20139. In addition, "2013" also occurs as the end of one number followed by the beginning of the next number. The different possibilities are:

2|013 does not occur, since no numbers starts with digit '0'.

20|13 occurs 11 times: 1320|1321 and 13020|13021 to 13920|13921.

201|3 occurs only once: 3201|3202, because no numbers larger than 30 000 were written down.

It is easy to verify that "2013" does not occur as a combination of three consequtive numbers. Therefore, "2013" occurs a total of $13 + 11 + 1 = 25$ times in the sequence of digits.