First Round
Dutch Mathematical Olympiad

Friday, January 30, 2009

Solutions

A1. Ella has done three tests. During the first one, she answered 60% of the 25 questions correctly, during the second one, she answered 70% of the 30 questions correctly, and during the last one, she answered 80% of the 45 questions correctly. Now if we merge these three tests together to form one of 100 questions, what is the percentage of these 100 questions that Ella answered correctly? (C) 72%

Solution 60% of 25 is 15; 70% of 30 is 21; and 80% of 45 is 36. So in total, Ella answered $15 + 21 + 36 = 72$ of the 100 questions correctly. $\square$

A2. How many of the integers from 10 to 99 (10 and 99 included) have the property that the sum of their digits is equal to the square of an integer? (An example: The sum of the digits of 27 is equal to $2 + 7 = 9 = 3^2$.) (E) 17

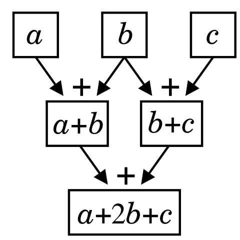
Solution We check how many of these numbers have sum of digits equal to 1, 2, etc. There is 1 number with sum 1 (being 10); there are 2 with sum 2 (being 20 and 11); etc.; 9 with sum 9 (being $90, 81, \ldots, 18$); also, 9 with sum 10 (being $91, 82, \ldots, 19$); etc.; and finally, 1 with sum 18 (being 99); see the table below. Then the sum of digits is a square of an integer (i.e. 1, 4, 9 or 16) in $1+4+9+3 = 17$ of the 90 cases.

sum:	**1**	2	3	**4**	5	6	7	8	**9**	10	11	12	13	14	15	**16**	17	18
number:	**1**	2	3	**4**	5	6	7	8	**9**	9	8	7	6	5	4	**3**	2	1

$\square$

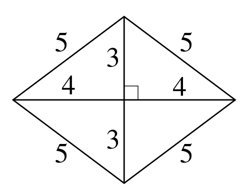
A3. Ronald throws three dice. These dice look just like ordinary dice, but their faces are numbered differently. The first die has the numbers 1, 1, 2, 2, 3 and 3 on it. The second die has the numbers 2, 2, 4, 4, 6 and 6 on it. And the third die has the numbers 1, 1, 3, 3, 5 and 5 on it. He then adds up the three numbers he gets from rolling the three dice. What is the probability that the resulting number is odd? (B) $\frac{1}{3}$

Solution Note that the second die only has even numbers on it, and that the third die only has odd numbers on it. So essentially the question is to find the probability that rolling the first die gives an *even* number. Since 2 of the 6 numbers on this die are even, this probability is equal to $\frac{2}{6} = \frac{1}{3}$. $\square$



A4. Three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are placed in the three squares in the top of the figure to the right, after which the numbers are added as described in said figure. We call Max the highest number that can appear in the bottom square, and Min the lowest number that can appear there. What is the value of Max − Min? (D) 26

Solution Let's say we put a, b, and c in the top three squares. Then the result in the bottom square is $a + 2b + c$. So we can maximize this result by making first b, then a and c as large as possible. Taking $b = 9, a = 8$ and $c = 7$ then yields 33 as result. In the same way, we can minimize the result by making first b, then a and c as small as possible. Taking $b = 1$, $a = 2$, $c = 3$ yields 7 as result. The difference between these numbers is $33 - 7 = 26$. $\square$



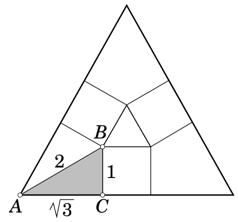
A5. The lengths of the diagonals of a rhombus have a ratio of 3 : 4. (A rhombus is an equilateral quadrilateral.) The sum of the lengths of the diagonals is 56. What is the perimeter of this rhombus? (A) 80

Solution Note that the diagonals have lengths $\frac{3}{7} \cdot 56 = 24$ and $\frac{4}{7} \cdot 56 = 32$. So the halves of diagonals have lengths $12 = 3 \cdot 4$ and $16 = 4 \cdot 4$. So the rhombus is 4 times larger than the rhombus in the figure, which consists of four triangles with sides 3, 4 and 5. Hence the sides of the original rhombus have length $4 \cdot 5 = 20$, and thus the perimeter has length $4 \cdot 20 = 80$. $\square$

A6. Wouter is traveling by foot from his home to the fitness center. He also could have chosen to travel by bike, in which case he would travel 7 times as fast. But he left his bike at home. After walking for 1 km, continuing to walk would take just as long as walking back to get his bike, and then travel further by bike. By then, what is the distance in km to the fitness center? (E) $\frac{4}{3}$

Solution Let x be said distance and let us suppose that he has spent a quarter of an hour walking by then. Then continuing walking will take him x quarters of an hour. On the other hand, if he decides to walk back home to pick up his bike, he'll first have to spend one quarter of an hour to get back, and then $\frac{1+x}{7}$ quarters of an hour by bike; since he travels 7 times faster that way. Then we have $x = 1 + \frac{1+x}{7}$, so $7x = 7 + (1 + x)$, or $6x = 8$. We deduce that $x = \frac{8}{6} = \frac{4}{3}$. Taking for 'quarter of an hour' any other time unit will give us the same result. $\square$

A7. On the sides of an equilateral triangle, we draw three squares. The sides of these squares that are parallel to the sides of the triangle are extended until they intersect. These three intersections form another equilateral triangle. Suppose that the length of a side of the original triangle is equal to 1. What is the length of a side of the large equilateral triangle? (D) $1+2\sqrt{3}$

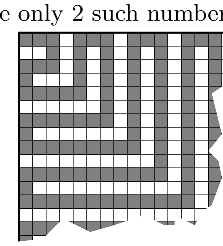


Solution In $\triangle ABC$, $\angle A$ is half of $60°$, so $30°$. Also, $\angle C$ is a right angle, so $\triangle ABC$ is a $30°$-$60°$-$90°$-triangle, where $|BC| = 1$. So it's half of an equilateral triangle with sides 2: $|AB| = 2$. Now we calculate $|AC|$ with the Theorem of Pythagoras: $|AC| = \sqrt{2^2 - 1^2} = \sqrt{3}$. So the required length is $\sqrt{3}+1+\sqrt{3}$. □

A8. Consider all four-digit numbers where each of the digits 3, 4, 6 and 7 occurs exactly once. How many of these numbers are divisible by 44? (A) 2

Solution Suppose that n, having digits a, b, c and d (so $n = 1000a + 100b + 10c + d$) is divisible by 44. Then it is also divisible by 11. Since the number $m = 1001a + 99b + 11c$ is also divisible by 11, so is $m - n$. Hence $m - n = a - b + c - d$ is a multiple of 11. But this number is at most the sum of the two highest digits, minus the sum of the lowest two, so $13 - 7 = 6$, and in the same way, we see that this number is at least -6. So it has to be equal to 0. Thus $a + c = b + d$, and since the sum of the digits is 20, we have $a + c = b + d = 10$. First suppose that $d = 4$. Then $b = 6$ so we get two possibilities for n, namely 3674 and 7634. But neither of these is divisible by 4, let alone by 44. Now suppose that $d = 6$, then we have $b = 4$, and in this case we get 3476 and 7436, both of which are divisible by 44. Finally, note that since n is divisible by 4, d must be even. We deduce that we have only 2 such numbers that are divisible by 44.
Alternative solution Check all 24 possibilities, or just the 12 even ones, or only the 6 multiples of 4. □

B1. On a sheet of paper, a grid of 101 by 101 white squares is drawn. A chain is formed by coloring squares grey as shown in the figure to the right. The chain starts in the upper left corner and goes on until it cannot go on any further. Only part of the grid is shown. In total, how many squares are colored grey in the original grid of 101 by 101 squares? 5201



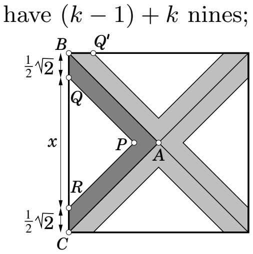
Solution We can subdivide this grid of 101^2 squares as follows. In the upper left corner, we have one (grey) square, then two L-shaped pieces, one having 3 squares (one of which grey), the other having 5 squares (all of which grey). Then we have two more L-shaped pieces, one having 7 squares (one of which grey), the other having 9 squares (all of which grey), etc. Of the last two L-shaped pieces, the first one has 199 squares (one of which grey), and the second one has 201 squares (all of which grey). We have 50 pairs of L-shapes in total, so the total number of grey squares is $1 + (1+5) + (1+9) + (1+13) + \ldots + (1+201) = 1 + (6+10+14+\ldots+202) = 1 + \frac{1}{2}\cdot 50 \cdot (6+202) = 5201$.
Alternative solution In each pair of these L-shapes, there are 4 more grey squares than white squares. So there are $50 \cdot 4 = 200$ more grey squares than white squares in the $101^2 - 1 = 10200$ squares contained in the 50 pairs of L-shapes, so we have 5000 white squares and 5200 grey ones. Since the upper left square is grey, in total, we have 5201 grey squares. □

B2. The integer N consists of 2009 consecutive nines. A computer calculates $N^3 = (99999\ldots99999)^3$. How many nines does the number N^3 contain in total? 4017

$$\underbrace{99999\ldots99999}_{2009\times}$$

Solution $9^3 = 729$; $99^3 = 970299$; $999^3 = 997002999$. It seems to be the case that in general, the third power of a number n consisting of k consecutive nines takes the following form: first $k-1$ nines; then a 7; then $k-1$ zeroes; then a 2; and finally k nines. To prove this, we write $n = 10^k - 1$. Indeed: $(10^k-1)^3 = 10^{3k} - 3\cdot 10^{2k} + 3\cdot 10^k - 1 = 10^{2k}(10^k - 3) + (3\cdot 10^k - 1)$. The number $10^k - 3$ can be written as $999\ldots997$ with $k-1$ nines. Multiplied with 10^{2k} this gives a number that ends in $2k$ zeroes. Adding $3\cdot 10^k - 1$ to this number, the last $k+1$ zeroes are replaced with $2999\ldots999$ with k nines. So in total, we have $(k-1)+k$ nines; in our case, $k = 2009$, so we have 4017 nines. □

B3. Using a wide brush, we paint the diagonals of a square tile, as in the figure. Exactly half of the area of this tile is covered with paint. Given that the width of the brush is 1, as indicated in the figure, what is the length of the side of the tile? $2+2\sqrt{2}$



Solution We only need to look at a quarter of the tile: $\triangle ABC$. The area of $\triangle PQR$ is half of the area of $\triangle ABC$. The triangles are similar, so corresponding sides have a ratio of $1 : \sqrt{2}$, so $|QR| : |BC| = 1 : \sqrt{2}$. Now we calculate $|BQ|$ using the Theorem of Pythagoras in $\triangle BQQ'$: $2|BQ|^2 = |BQ'|^2 + |BQ|^2 = 1^2$, so $|BQ| = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$. Now let us write x for $|QR|$. Then we find $x + \sqrt{2} = \sqrt{2}\cdot x$, so $x(\sqrt{2}-1) = \sqrt{2}$ or equivalently, $x = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = 2+\sqrt{2}$. Hence $|BC| = x + \sqrt{2} = 2 + 2\sqrt{2}$ (or $|BC| = \sqrt{2}\cdot x = \sqrt{2}\cdot(2+\sqrt{2}) = 2\sqrt{2}+2$). □

B4. Determine a triplet of integers (a,b,c) satisfying: $a+b+c = 18$; $a^2+b^2+c^2 = 756$ and $a^2 = bc$.
$(a,b,c) = (-12, 6, 24)$ of $(a,b,c) = (-12, 24, 6)$ (one answer is enough)

Solution We calculate $(b+c)^2$ in two different ways. $(b+c)^2 = (18-a)^2 = 324 - 36a + a^2$ and $(b+c)^2 = b^2 + 2bc + c^2 = (756 - a^2) + 2a^2$. So $a^2 - 36a + 324 = a^2 + 756$, or $-36a = 756 - 324 = 432$, so $a = -12$. Substituting this in the first and in the last equation, we obtain the equations $b + c = 30$ and $bc = 144$. Trying some divisors of $144 = 12^2$, we then should be able to find a solution. Or we can just substitute $c = 30 - b$ in the last equation, yielding the quadratic equation $b(30-b) = 144$, or equivalently $b^2 - 30b + 144 = 0$. We can factorize this as $(b-6)(b-24) = 0$ (or we can use the *abc*-formula) to see that we have two solutions $b = 6$ (and $c = 24$) or $b = 24$ (and $c = 6$).
Alternative solution Just as above, we see that $a = -12$. Then by substituting this in all three equations, we see that $b+c = 30$, $b^2 + c^2 = 756 - 144 = 612$ and $bc = 144$. Combining the last two equations yields $(b-c)^2 = b^2 + c^2 - 2bc = 612 - 2\cdot 144 = 324$, or equivalently $b - c = \pm\sqrt{324} = \pm 18$. Adding the equations $b + c = 30$ and $b - c = -18$, we get $2b = (b+c) + (b-c) = 12$ so $b = 6$ (and $c = 24$). Adding the equations $b + c = 30$ and $b - c = 18$, we get $2b = (b+c) + (b-c) = 48$ so $b = 24$ (and $c = 6$). □