The 47th Dutch Mathematical Olympiad 2008

and the team selection for IMO 2009 Bremen

Contents

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Introduction

The Dutch Mathematical Olympiad consists of two rounds. The first round is held on the participating schools and consists of eight multiple choice questions and four open questions (see page 1–4). Students get two hours to work on the paper. On January 25, 2008, in total 3004 students of 201 secondary schools participated in this first round.

The best students are invited for the second round. For the first time we set different thresholds for students from different grades, in order to stimulate young students to participate. Those students from grade 5 (4, ≤ 3) that scored 26 (22, 18) points or more on the first round (out of a maximum of 36 points) were invited to the second round. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited.

As a new initiative, we organized training sessions at six universities in the country for the 143 students who had been invited for the second round in September 2009. Former Dutch IMO-participants are involved at each of the universities.

From those 143, in total 123 participated in the second round on September 12th, 2008 at the Eindhoven University of Technology. This final round contains five problems for which the students have to give extensive solutions and proofs. They have three hours for the paper (see page 5–7). After the prices had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 47th edition 2008. In 2011 we will have our 50th edition.

The 31 most outstanding candidates of the Dutch Mathematical Olympiad 2008 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June.

In May, ten of them were invited to participate to the first Benelux Mathematical Olympiad, held in Bergen op Zoom, the Netherlands (see page 8–12). We think this has been a very stimulating experience for our students.

In June, 24 out of these 31 candidates were left. Out of those, the team was selected by a final selection test on June 13, 2009 (see page 13–16). The team will have a training camp from July 6 until July 13, together with the team from New Zealand.

In the meantime, a new edition of the Dutch Mathematical Olympiad had started. In October 2008, at the VU University Amsterdam we organized the first Junior Mathematical Olympiad for the winners of the popular Kangaroo math contest (see page 17–20). The 30 best students of grade 2, grade 3 and grade 4 were invited. We hope all of them will enjoy those problems so much, that they will participate to the first round in the years that come.

The new first round took place at participating schools on January 30, 2009 (see page 21–24). We are very happy to see that the number of schools as well as the number of participants has increased again: 230 schools and 4379 participants, a new record since ages!

The Dutch team for IMO 2009 Bremen consists of

- Wouter Berkelmans (18 y.o., participated in IMO 2006, 2007 as well)
- Raymond van Bommel (17 y.o., participated in IMO 2007, 2008 as well)
- Harm Campmans (17 y.o.)
- Saskia Chambille (18 y.o.)
- David Kok (16 y.o.)
- Maarten Roelofsma (18 y.o., participated in IMO 2008 as well)

As a promising young student, we bring

- Merlijn Staps (14 y.o.)

The team is coached by

- Quintijn Puite (team leader), Eindhoven University of Technology
- Birgit van Dalen (deputy leader), Leiden University

The Dutch delegation for IMO 2009 Bremen further consists of

- Wim Berkelmans (member AB, observer A), VU University Amsterdam
- Hans van Duijn (observer A), Eindhoven University of Technology
- Tom Verhoeff (observer A), Eindhoven University of Technology
- Gerhard Wöginger (observer A), Eindhoven University of Technology
- Ronald van Luijk (observer A), Leiden University
- Jelle Loois (observer B), Ortec
- Anick van de Craats (observer C), Netherlands Forensic Institute
- Lidy and Theo Wesker (observer C), University of Amsterdam
- Karst Koymans, University of Amsterdam
- Freek van Schagen, VU University Amsterdam
- Rob Wieleman, Movisie Utrecht
- Wendoline Timmerman, Ministry of Education, Culture and Science

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We are grateful to Jinbi Jin for the translation into English of most of the problems and the solutions. We also thank Lieneke Notenboom-Kronemeijer for her useful remarks concerning the formulation of the First Round 2009 paper.

First Round
Dutch Mathematical Olympiad

Friday, January 25, 2008

Problems

- Time available: 2 hours.
- The A-problems are multiple choice questions. Only one of the five options given is correct. Please state clearly which letter precedes the correct solution. Each correct answer is worth 2 points.
- The B-problems are open questions; the answers to these questions are a number, or numbers. Each correct answer is worth 5 points. Please work accurately, since an error in your calculations may cause your solution to be considered incorrect and then you won't get points at all for that question. Please give your answers exactly, for example $\frac{11}{81}$ or $2 + \frac{1}{2}\sqrt{5}$ or $\frac{1}{4}\pi + 1$.
- You are not allowed to use calculators and formula sheets; you can only use a pen, a compass and a ruler or set square. And your head, of course.
- This is a competition, not an exam. The main thing is that you have fun solving unusual mathematical problems. Good luck!

A-problems

A1. Alex, Birgit, Cedric, Dion and Ersin all write their names on a sheet of paper, and they put those five sheets into a large box. They each take one sheet out of the box at random. Now it turns out that Birgit has Alex' sheet, Cedric has Dion's and Dion has Ersin's. Also, Ersin doesn't have Cedric's sheet. Whose sheet does Alex have?

A) Alex' B) Birgit's C) Cedric's D) Dion's E) Ersin's

A2. In a magic 3×3 square, the three row sums, the three column sums and the two diagonal sums are all equal to each other. (A *row sum* being the sum of the numbers on a certain row, etc.) In the magic 3×3 square shown here three numbers have already been filled in. What number must be filled in instead of the question mark?

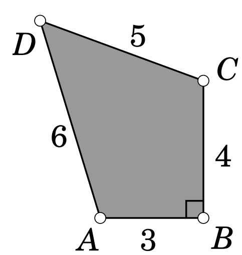
		7
?		
	10	3

A) 2 B) 4 C) 6 D) 8 E) 9

A3. Calculating $6 \times 5 \times 4 \times 3 \times 2 \times 1$ yields 720. How many divisors does 720 have? (A *divisor* of an integer n is a positive integer by which n is divisible. For example: the divisors of 6 are 1, 2, 3 and 6; the divisors of 11 are 1 and 11.)

A) 6 B) 8 C) 20 D) 30 E) 36

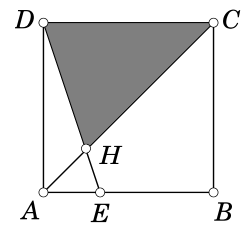
A4. Of a quadrilateral $ABCD$, we know that $|AB| = 3$, $|BC| = 4$, $|CD| = 5$, $|DA| = 6$ en $\angle ABC = 90°$. ($|AB|$ stands for the length of segment AB, etc.) What is the area of quadrilateral $ABCD$?



A) 16 B) 18 C) $18\frac{1}{2}$ D) 20 E) $6 + 5\sqrt{11}$

A5. How many five-digit numbers (like 12345 or 78000; the first digit must be non-zero) are there that end on a 4 and that are divisible by 6?

A) 1500 B) 2000 C) 3000 D) 7500 E) 8998



A6. We have a square $ABCD$ with $|AB| = 3$. On AB, there is a point E such that $|AE| = 1$ and $|EB| = 2$. AC and DE intersect in H. What is the area of triangle CDH?

A) $\frac{9}{8}$ B) 2 C) $\frac{21}{8}$ D) 3 E) $\frac{27}{8}$

A7. The seven blocks $\boxed{\text{S}}\boxed{\text{E}}\boxed{\text{T}}\boxed{\text{T}}\boxed{\text{E}}\boxed{\text{E}}\boxed{\text{S}}$ are shuffled. For example, you can get $\boxed{\text{E}}\boxed{\text{E}}\boxed{\text{E}}\boxed{\text{S}}\boxed{\text{S}}\boxed{\text{T}}\boxed{\text{T}}$ or $\boxed{\text{T}}\boxed{\text{E}}\boxed{\text{S}}\boxed{\text{E}}\boxed{\text{T}}\boxed{\text{E}}\boxed{\text{S}}$.
How many different “words” of length 7 can we get this way? (Any combination of the 7 letters counts as word.)

A) 210 B) 420 C) 840 D) 1260 E) 5040

A8. How many distinct real solutions does the equation $\left((x^2-2)^2-5\right)^2 = 1$ have?

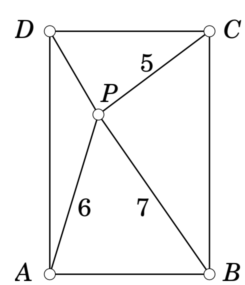
A) 4 B) 5 C) 6 D) 7 E) 8

B-problems

B1. We number both the rows and the columns of an 8×8 chessboard with the numbers 1 to 8. A number of grains is placed onto each square, in such a way that the number of grains on a certain square equals the product of its row and column numbers. How many grains are there on the entire chessboard?

B2. We take 50 distinct integers from the set $\{1, 2, 3, \ldots, 100\}$, such that their sum equals 2900. What is the minimal number of *even* integers amongst these 50 numbers?

B3. For a certain x, we have $x + \frac{1}{x} = 5$. Define $n = x^3 + \frac{1}{x^3}$. It turns out that n is an integer.
Calculate n. (Give your answer using decimal notation.)



B4. Inside a rectangle $ABCD$, there is a point P with $|AP| = 6$, $|BP| = 7$ and $|CP| = 5$. What is the length of segment DP?

First Round
Dutch Mathematical Olympiad

Friday, January 25, 2008

Solutions

A1. C) Cedric When we put the data into a table, we see that Birgit's and Cedric's sheets haven't been picked yet. Since Ersin didn't pick Cedric's sheet, he must have picked Birgit's. So Alex must have picked Cedric's sheet.

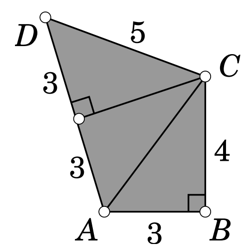
A	B	C	D	E
?	A	D	E	?

A2. B) 4 See the figure. From $F + 10 + 3 = F + D + 7$, we get $D = 6$. Then from $7 + E + 3 = C + D + E = C + 6 + E$, we can deduce that $C = 7 + 3 - 6 = 4$.

A	B	7
C	D	E
F	10	3

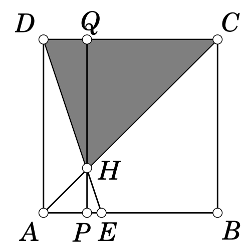
A3. D) 30 The number 720 only has the prime factors 2, 3 and 5. The prime factor 2 occurs four times (once in 2, twice in 4 and once in 6), the prime factor 3 twice (once in 3 and 6), and 5 just once. The divisors without any factors 3 or 5 are 1, 2, 4, 8 and 16. The divisors having one factor 3 and no factors 5 are 3, 6, 12, 24, 48. And the divisors having two factors 3 and no factors 5 are 9, 18, 36, 72 and 144. So 720 has 15 divisors that do not have factors 5. Multiplying all of these divisors by 5 gives us the other 15 divisors, which makes 30 in total.
Alternative solution: Every divisor of $720 = 2^4 \times 3^2 \times 5^1$ can be written as $2^a \times 3^b \times 5^c$ with 5 possibilities for a (being 0 to 4), 3 possibilities for b (being 0 to 2) and 2 for c (being 0 and 1). So we conclude that 720 has $5 \times 3 \times 2 = 30$ divisors.

A4. B) 18 According to Pythagoras' Theorem, we have $|AC| = 5$. So triangle ACD is isosceles with base AD. In this triangle, the altitude from C divides the triangle into two triangles with sides 3, 4 and 5, and we can divide quadrilateral $ABCD$ in three triangles with sides 3, 4 and 5. So its area must be equal to $3 \times 6 = 18$.

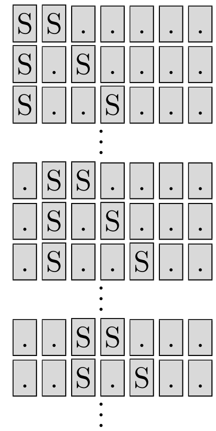


A5. C) 3000 If x is a positive multiple of 6 that ends with a 4, then the next multiples of 6 end with a 0 ($x + 6$), a 6 ($x + 12$), a 2 ($x + 18$), an 8 ($x + 24$), a 4 ($x + 30$), so the next multiple of 6 that ends with a 4 is $x + 30$. So any 30 consecutive positive integers must contain exactly one integer with the desired properties. How many such integers lie between 10000 and 99999? Since we have 90000 consecutive positive integers, we find $90000 \div 30 = 3000$ such integers amongst them.

A6. E) $\frac{27}{8}$ Draw a line through H parallel to AD, and let P and Q be the intersections of this line with AB and CD, respectively. Now we have $|HP| : |HQ| = |AE| : |CD| = 1 : 3$, so $|HQ| = \frac{3}{4} \times |PQ| = \frac{3}{4} \times 3 = \frac{9}{4}$. So the area of triangle CDH equals $\frac{1}{2} \times 3 \times \frac{9}{4} = \frac{27}{8}$.



A7. A) 210 We have $6+5+4+3+2+1=21$ (or $\binom{7}{2}$) possibilities to arrange the S-blocks onto the 7 places (see figure). For each choice, we have $4+3+2+1=10$ (or $\binom{5}{2}$) possibilities to arrange the T-blocks on the remaining 5 places; after which the positions of the E-blocks are determined. So we have $21\times 10=210$ possibilities.



Alternative solution: If all the blocks were different, we would have got 7! possibilities. But the three E-blocks aren't different, so we end up counting each word 3! times this way. Similarly for the two S-blocks and the two T-blocks. So we find $\frac{7!}{3!\times 2!\times 2!}=\frac{7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}\times 4}=7\times6\times5=$ 210 different words.

A8. B) 5 This equation is equivalent to

$$(x^2-2)^2-5=1 \quad \text{or} \quad (x^2-2)^2-5=-1.$$

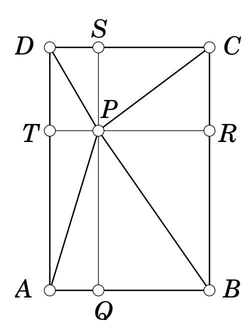
The first is equivalent to $x^2-2=\sqrt{6}$ of $x^2-2=-\sqrt{6}$, with 2 and 0 solutions respectively (since $-\sqrt{6}+2<0$).
The latter is equivalent to $x^2-2=2$ of $x^2-2=-2$, with 2 and 1 solution(s) respectively. So we have $2+0+2+1=5$ solutions in total.

B1. 1296 In the first column, we have, successively, $1\times1, 1\times2, 1\times3, \ldots, 1\times8$ grains.
So, in the first column: $1\times(1+2+3+4+5+6+7+8)$.
In the second column: $2\times(1+2+3+4+5+6+7+8)$.
In the third column is: $3\times(1+2+3+4+5+6+7+8)$.
$\vdots$
Finally, in the eighth column: $8\times(1+2+3+4+5+6+7+8)$.
So, in total: $(1+2+3+4+5+6+7+8)\times(1+2+3+4+5+6+7+8)$.
Since $1+2+3+4+5+6+7+8=\frac{1}{2}\times8\times(1+8)=36$, there are $36^2=1296$ grains on the board.

B2. 6 The 50 odd integers from the set $\{1,2,3,\ldots,100\}$ sum up to $\frac{1}{2}\times50\times(1+99)=$ 2500, which is still 400 short of 2900. Now exchange the smallest odd integers for the largest even integers, in pairs, since 400 is even. First exchanging 1 and 3 for 100 and 98 makes the sum equal to 2694. The next step gives us $2694-5-7+96+94=2872$. Which is still less than 2900, so we require another exchange. Now exchanging 9 and 11 for 20 and 28 works, making the sum 2900 with 6 even integers, showing along the way that that is the minimal number of even integers we need to do so.

B3. 110 We know that x must satisfy $x+\frac{1}{x}=5$, so $x^2-5x+1=0$, from which follows that $x=x_{1,2}=\frac{5\pm\sqrt{21}}{2}$. Note that $x_1x_2=1$. Now we have $x^3+x^{-3}=x_1^3+x_2^3=\frac{1}{8}\left(\left(5+\sqrt{21}\right)^3+\left(5-\sqrt{21}\right)^3\right)=$ $\frac{1}{8}\left(\left(5^3+3\cdot5^2\cdot\sqrt{21}+3\cdot5\cdot\sqrt{21}^2+\sqrt{21}^3\right)+\left(5^3-3\cdot5^2\cdot\sqrt{21}+3\cdot5\cdot\sqrt{21}^2-\sqrt{21}^3\right)\right)=$ $\frac{2}{8}\left(5^3+3\cdot5\cdot21\right)=110$.
Alternative solution: From $(x+\frac{1}{x})^3=x^3+3x^2(\frac{1}{x})+3x(\frac{1}{x})^2+(\frac{1}{x})^3=x^3+3x+\frac{3}{x}+\frac{1}{x^3}$ we can deduce that $x^3+\frac{1}{x^3}=(x+\frac{1}{x})^3-3(x+\frac{1}{x})=5^3-3\times5=110$.

B4. $2\sqrt{3}$ Let Q, R, S, T be the orthogonal projections of P on AB, BC, CD, DA, respectively. Then we have
$|AQ|^2+|QP|^2=36$ and $|BQ|^2+|SP|^2=25$ (since $|BQ|=|CS|$),
so $|AQ|^2+|QP|^2+|BQ|^2+|SP|^2=61$. We also have $|BQ|^2+|QP|^2=49$.
So $|DS|^2+|SP|^2=|AQ|^2+|SP|^2=61-49=12$ and $|DP|=\sqrt{12}=2\sqrt{3}$.



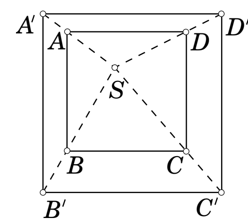
Second Round
Dutch Mathematical Olympiad

Friday, September 12, 2008
Eindhoven University of Technology

Problems

- Available time: 3 hours.
- Writing down just the answer itself is not sufficient; you also need to describe the way you solved the problem.
- Usage of calculators and formula sheets are not allowed; you are only allowed to use a pen, a compass and a ruler or set square. And your common sense of course.
- Please write the solutions of each problem on a different sheet of paper. Good luck!

1. Suppose we have a square $ABCD$ and a point S in the interior of this square. Under homothety with centre S and ratio of magnification $k > 1$, this square becomes another square $A'B'C'D'$.
Prove that the sum of the areas of the two quadrilaterals $A'ABB'$ and $C'CDD'$ are equal to the sum of the areas of the two quadrilaterals $B'BCC'$ and $D'DAA'$.

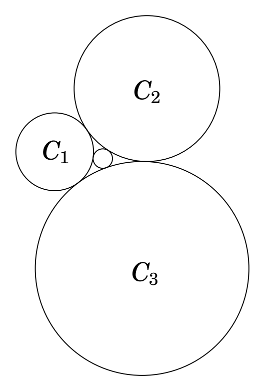


2. Find all positive integers (m, n) such that

$$3 \cdot 2^n + 1 = m^2.$$

3. Suppose that we have a set S of 756 arbitrary integers between 1 and 2008 (1 and 2008 included).
Prove that there are two distinct integers a and b in S such that their sum $a + b$ is divisible by 8.

4. Three circles C_1, C_2, C_3, with radii 1, 2, 3 respectively, are externally tangent. In the area enclosed by these circles, there is a circle C_4 which is externally tangent to all three circles.
Find the radius of C_4.



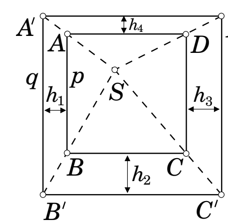
5. We're playing a game with a sequence of 2008 non-negative integers. A *move* consists of picking a integer b from that sequence, of which the neighbours a and c are positive. We then replace a, b and c by $a - 1$, $b + 7$ and $c - 1$ respectively. It is not allowed to pick the first or the last integer in the sequence, since they only have one neighbour.
If there is no integer left such that both of its neighbours are positive, then there is no move left, and the game ends.
Prove that the game always ends, regardless of the sequence of integers we begin with, and regardless of the moves we make.

Second Round
Dutch Mathematical Olympiad

Friday, September 12, 2008
Eindhoven University of Technology

Solutions

1. Let $p = |AB|$ and $q = |A'B'|$, so $q = k \cdot p$. Note that the sides AB and $A'B'$ are parallel, because of the homothety. Hence quadrilateral $A'ABB'$ is a trapezium, so its area equals the sum of the area of the triangles $\triangle ABB'$ and $\triangle A'B'A$, so it is equal to $\frac{1}{2} \cdot |AB| \cdot h_1 + \frac{1}{2} \cdot |A'B'| \cdot h_1 = \frac{p+q}{2} h_1$, where h_1 is the distance between the parallel lines AB and $A'B'$. Similarly, we see that the area of quadrilateral $C'CDD'$ is equal to $\frac{p+q}{2} h_3$, where h_3 is the distance between CD and $C'D'$. Hence the area of the two trapezia together is $\frac{p+q}{2}(h_1 + h_3)$. In the same way, we see that the area of the two trapezia $A'ADD'$ and $B'BCC'$ together is equal to $\frac{p+q}{2}(h_2 + h_4)$. Now note that $h_1 + h_3 = q - p = h_2 + h_4$, so the two areas are equal.



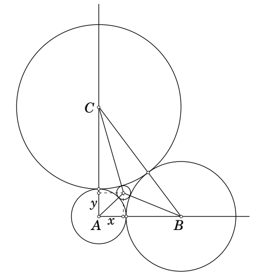
2. We can rewrite the equation as $3 \cdot 2^n = (m-1)(m+1)$. Since $n > 0$, we see that $3 \cdot 2^n$ is an even number, so at least one of $m-1, m+1$ is even as well. Hence they're both even. As the factors $m-1$ and $m+1$ differ by 2, they can't both contain multiple factors of 2. Hence one of these factors contains exactly one factor 2. This factor either contains the factor 3 as well, or it doesn't. So it must be equal to either 2 or 6. The other factor then must differ from this one by exactly 2. If this factor is equal to 2, then the other one has to be either 0 or 4. Since none of these contain a factor 3, none of these solve the equation. If this factor is equal to 6, then the other one has to be either 4 or 8, both of which yield a solution, as they are powers of 2. Hence there are exactly two solutions, given by $(m, n) = (5, 3)$ and $(m, n) = (7, 4)$.

3. We divide the integers 1 up to 2008 amongst eight distinct subsets V_1 to V_8, where V_i is the subset consisting of the 251 integers of the form $8k + i$ with $0 \leq k \leq 250$. So we have

$$V_1 = \{1, 9, \ldots, 2001\}, \quad \ldots, \quad V_8 = \{8, 16, \ldots, 2008\}.$$

The union of these subsets consists of the positive integers up to 2008.
Now suppose for a contradiction that there are no two integers a, b such that $a + b$ is divisible by 8. Then let us consider the distribution of S among the V_i. Note that the sum of two multiples of 8 is again a multiple of 8, so we see that V_8 then cannot contain more than 1 element of S. Since the sum of two integers from V_4 is a multiple of 8, V_4 cannot contain more than 1 element of S either. Next, we see that the sum of a integer from V_1 and one from V_7, is a multiple of 8, so at least one of them contains no elements of S. Similarly, we see that at least one of V_2 and V_6, and at least one of V_3 and V_5 contain no elements of S at all. Now note that every V_i contains at most 251 elements of S. Hence V_1, V_2, V_3, V_5, V_6, V_7 together contain at most $3 \cdot 251 = 753$ elements of S. So the sets V_1 to V_8 together contain at most 755 elements of S, which is a contradiction, since S consists of 756 elements, and each element is contained in a certain V_i.

4. Let A, B, C be the centres of the circles C_1, C_2, C_3 respectively. The triangle formed by these three centres, has sides $|AB| = 3$, $|AC| = 4$ and $|BC| = 5$, hence it is a right-angled triangle. Now choose the x- and the y-axis in such a way that $A = (0,0)$, $B = (3,0)$ and $C = (0,4)$. Let r and $M = (x,y)$ be the radius and centre of C_4, respectively. Then we see that $|AM| = r+1$, $|BM| = r+2$ and $|CM| = r+3$. This implies the three following equations with three unknowns.



$$r^2 + 2r + 1 = |AM|^2 = x^2 + y^2 \tag{1}$$
$$r^2 + 4r + 4 = |BM|^2 = (3-x)^2 + y^2 = x^2 - 6x + 9 + y^2 \tag{2}$$
$$r^2 + 6r + 9 = |CM|^2 = x^2 + (4-y)^2 = x^2 + y^2 - 8y + 16 \tag{3}$$

Taking the difference of (1) and (2), we see that $6x - 9 = -2r - 3$, so $6x = 6 - 2r$, hence $x = \frac{3-r}{3}$. Taking the difference of (1) and (3), we see that $8y - 16 = -4r - 8$, so $8y = 8 - 4r$, hence $y = \frac{2-r}{2}$. Substituting this in (1) yields $r^2 + 2r + 1 = x^2 + y^2 = \frac{(3-r)^2}{9} + \frac{(2-r)^2}{4} = \frac{9-6r+r^2}{9} + \frac{4-4r+r^2}{4}$, thus $\frac{23}{36}r^2 + \frac{11}{3}r - 1 = 23\left(\frac{r}{6}\right)^2 + 22\left(\frac{r}{6}\right) - 1 = 0$. Substituting $p = \frac{r}{6}$, we then see that $0 = 23p^2 + 22p - 1 = (23p-1)(p+1)$. This gives the two possible solutions $\frac{r}{6} = p = -1$ and $23 \cdot \frac{r}{6} = 23p = 1$. But since r has to be positive, we see that r has to be equal to $\frac{6}{23}$. (We can deduce from this that $M = (x,y) = (\frac{21}{23}, \frac{20}{23})$.)

5. Consider an arbitrary sequence $n_1, n_2, \ldots, n_{2008}$ and an arbitrary sequence of moves. The first integer n_1 is reduced by 1 every time we pick $b = n_2$, since n_1 is one of the neighbours of n_2. If we pick $b = n_k$, where $k > 2$, then we see that n_1 remains unchanged. Hence we can pick $b = n_2$ at most n_1 times. So in our sequence of moves, either there is a last time that the move $b = n_2$ occurs, or that move doesn't occur at all. In the former case, we consider the sequence of moves following that move, and in the latter case, we simply consider the sequence of all moves. In this new sequence of moves, we never pick $b = n_2$. So all we do in this new sequence, is picking $b = n_3$ up to $b = n_{2007}$. Let n_2 now be the value of the second integer at the beginning of this new sequence (this integer need not be the same as before, since every time we've picked $b = n_2$ until then, it is increased by 7, and every time we've picked $b = n_3$ until then, it is reduced by 1). In the new sequence, n_2 is reduced by 1 every time we pick $b = n_3$, and remains unchanged if we pick any other integer (i.e. $b = n_4$ up to $b = n_{2007}$). Hence we can conclude that in the new sequence, either there is a last time that the move $b = n_3$ occurs, or this move doesn't occur at all. So from that point onward, we only pick the integers $b = n_4$ up to $b = n_{2007}$. We can repeat this argument to see that, from some point onward, only $b = n_{2007}$ is picked, and that there either is a last time that the move $b = n_{2007}$ occurs, or this move doesn't occur at all, indicating that our sequence of moves ends at some point. □

Variant Suppose that there exists an infinite sequence of moves. By the pigeonhole principle, there is a move $b = n_k$ that occurs infinitely often. Now let k be the smallest integer for which the move $b = n_k$ occurs infinitely often. Then we see that the integer $b = n_{k-1}$ is only picked finitely often, hence is increased only finitely often. But this integer is decreased infinitely often; namely for every time we pick $b = n_k$. This yields our contradiction. □

Alternative solution For each sequence of 2008 integers $n_1, n_2, \ldots, n_{2008}$, we compute the weighted sum $S = 7 \cdot n_1 + 7^2 \cdot n_2 + 7^3 \cdot n_3 + \cdots + 7^{2007} \cdot n_{2007} + 7^{2008} \cdot n_{2008}$. If we replace a, b, c in this sequence by $a-1, b+7, c-1$, where $b = n_k$ for a certain integer k such that $2 \le k \le 2007$, then S becomes equal to $S - 7^{k-1} + 7 \cdot 7^k - 7^{k+1} = S - 7^{k-1}$, so every move reduces the value of S. On the other hand, we note that S is a sum of non-negative integers, hence is a non-negative integer itself. Writing down the new value of S after each move, we get a decreasing sequence of non-negative integers. Such a sequence can not be infinite. Hence the game ends after a finite number of moves.

1st BENELUX MATHEMATICAL OLYMPIAD
Bergen op Zoom (Netherlands)
May 9, 2009

Language: **English**

Problem 1. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that satisfy the following two conditions:

- $f(n)$ is a perfect square for all $n \in \mathbb{Z}_{>0}$;
- $f(m+n) = f(m) + f(n) + 2mn \quad$ for all $m, n \in \mathbb{Z}_{>0}$.

Problem 2. Let n be a positive integer and let k be an odd positive integer. Moreover, let a, b and c be integers (not necessarily positive) satisfying the equations

$$a^n + kb = b^n + kc = c^n + ka.$$

Prove that $a = b = c$.

Problem 3. Let $n \geq 1$ be an integer. In town X there are n girls and n boys, and each girl knows each boy. In town Y there are n girls, $g_1, g_2, \ldots, g_n$, and $2n-1$ boys, $b_1, b_2, \ldots, b_{2n-1}$. For $i = 1, 2, \ldots, n$, girl g_i knows boys $b_1, b_2, \ldots, b_{2i-1}$ and no other boys. Let r be an integer with $1 \leq r \leq n$. In each of the towns a party will be held where r girls from that town and r boys from the same town are supposed to dance with each other in r dancing pairs. However, every girl only wants to dance with a boy she knows. Denote by $X(r)$ the number of ways in which we can choose r dancing pairs from town X, and by $Y(r)$ the number of ways in which we can choose r dancing pairs from town Y.
Prove that $X(r) = Y(r)$ for $r = 1, 2, \ldots, n$.

Problem 4. Given trapezoid $ABCD$ with parallel sides AB and CD, let E be a point on line BC outside segment BC, such that segment AE intersects segment CD. Assume that there exists a point F inside segment AD such that $\angle EAD = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, and assume that K is different from I and J.
Prove that K belongs to the circumcircle of $\triangle ABI$ if and only if K belongs to the circumcircle of $\triangle CDJ$.

Time allowed: 4 hours and 30 minutes
Each problem is worth 7 points

1st BENELUX MATHEMATICAL OLYMPIAD
Bergen op Zoom (Netherlands)
May 9, 2009

Language: **English**

Solutions

Problem 1.

Solution 1. Let a be a positive integer satisfying $f(1) = a^2$. We will prove that $f(n) = na^2 + n(n-1)$ for all n by induction on n. For $n = 1$ it follows from the definition of a. Now suppose we have $f(n) = na^2 + n(n-1)$ for a certain positive integer n. Then by the second condition with $m = 1$, we have

$$f(n+1) = f(n) + f(1) + 2n = na^2 + n(n-1) + a^2 + 2n = (n+1)a^2 + n(n+1).$$

This completes the induction.

Suppose $a > 1$ and let p be a prime divisor of a. We know that $f(p) = pa^2 + p(p-1)$ is a square, and as it is obviously divisible by p, it must be divisible by p^2. Hence $a^2 + p - 1$ is divisible by p. But this is a contradiction, as a and p are both divisible by p.

We conclude $a = 1$ and $f(n) = n^2$ for all $n \in \mathbb{Z}_{>0}$. This function indeed satisfies the conditions. $\square$

Comment. Proving that $a = 1$ can be done in various ways. Here is another possibility. We have $f(n) = na^2 + n(n-1) = n(a^2-1) + n^2$ for all $n > 0$. Suppose $a > 1$, then $n = a^2 - 1 > 0$ and $f(a^2-1) = 2(a^2-1)^2$. As all prime factors in $(a^2-1)^2$ occur an even number of times, the prime factor 2 occurs an odd number of times in $f(a^2-1) = 2(a^2-1)^2$. Hence this is not a square, which contradicts the first condition.

Solution 2. Define $g(n) = f(n) - n^2$ for all $n \in \mathbb{Z}_{>0}$. Then we can rewrite the second condition as

$$g(m+n) + (m+n)^2 = g(m) + m^2 + g(n) + n^2 + 2mn,$$

hence $g(n)$ satisfies the functional equation

$$g(m+n) = g(m) + g(n) \quad \text{for all } m, n \in \mathbb{Z}. \tag{4}$$

Let $b = g(1) = f(1) - 1 \geq 0$. By setting $n = 1$ in (4) we find $g(m+1) = g(m) + b$, and hence by induction we have $g(n) = nb$ for all n. Therefore $f(n) = nb + n^2$ for all n. Suppose $b > 0$, then taking $n = b$ yields $f(b) = 2b^2$, which is not a square, contradicting the first condition.

We conclude $b = 0$ and $f(n) = n^2$ for all $n \in \mathbb{Z}_{>0}$. This function indeed satisfies the conditions. $\square$

Problem 2.

Solution 1. First suppose $a = b$. From $a^n + kb = b^n + kc$ it then follows that $b = c$, which means we are done. Similarly, we are done if $b = c$ or $c = a$. Now suppose $a \neq b$, $b \neq c$ en $c \neq a$. We will derive a contradiction.

From $a^n + kb = b^n + kc$ we find $a^n - b^n = k(c - b)$. In a similar way we find two more such equations, so we have

$$a^n - b^n = k(c - b), \qquad b^n - c^n = k(a - c), \qquad c^n - a^n = k(b - a). \tag{5}$$

Multiply these three equations and divide by $(a - b)(b - c)(c - a)$:

$$\frac{a^n - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = \frac{a^n - b^n}{b - c} \cdot \frac{b^n - c^n}{c - a} \cdot \frac{c^n - a^n}{a - b} = -k^3. \tag{6}$$

Note that the left-hand side is the product of three integers.

Suppose n is odd. Then x^n is a monotonically increasing function of x. So if $a < b$, then $a^n < b^n$, and so on. Hence $\frac{a^n - b^n}{a - b}$ is positive and the same holds for the other two factors at the left of (6). Contradiction with $-k^3 < 0$. We conclude that n is even.

Consider a, b and c modulo 2. According to the box principle, at least two of those are congruent, say $a \equiv b \mod 2$. The integer $\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}$ is the sum of n terms, and as $a \equiv b \mod 2$, all of these terms are even, or all of these terms are odd. In any case, the sum of the n terms is even. So the left-hand side of (6) is even, while the right-hand side is odd. This is a contradiction, which finishes the proof. $\square$

Solution 2. As in the first solution, we assume $a \neq b$, $b \neq c$ en $c \neq a$ and we find

$$a^n - b^n = k(c - b), \qquad b^n - c^n = k(a - c), \qquad c^n - a^n = k(b - a). \tag{5}$$

First consider the case that n is odd. Then x^n is a monotonically increasing function of x. So if $a < b$, then $a^n < b^n$, and so on. So suppose $a > b$, then we have $a^n > b^n$, hence $a^n - b^n > 0$. Hence by (5) we have $c - b > 0$. So $b^n - c^n < 0$ and therefore $a - c < 0$. From this we have $c^n - a^n > 0$, so $b - a > 0$, which contradicts the assumption. Similarly we get a contradiction if $a < b$.

Now assume n is even. Consider the equalities in (5) modulo 2. As k is odd, the first equality gives $a - b \equiv c - b \mod 2$, hence $a \equiv c \mod 2$. Now the integer $\frac{c^n - a^n}{c - a} = c^{n-1} + c^{n-2}a + \cdots + ca^{n-2} + a^{n-1}$ is the sum of n terms, and as $a \equiv c \mod 2$, all of these terms are even, or all of these terms are odd. In any case, the sum of the n terms is even. Let i be the exponent of the prime factor 2 in $c - a$. Then the exponent of the prime factor 2 in $c^n - a^n$ is at least $i + 1$. From the third equality in (5) we derive that the exponent of 2 in $b - a$ is at least $i + 1$ as well. Similarly to the previous argument, the exponent of 2 in $a^n - b^n$ must be at least $i + 2$, and the same holds for the exponent of 2 in $c - b$. Finally the exponent of 2 in $b^n - c^n$ and hence also in $a - c$ must be at least $i + 3$. This contradicts the definition of i. $\square$

Problem 3.

As $X(r)$ and $Y(r)$ are dependent on n, we will from now on denote them by $X_n(r)$ and $Y_n(r)$.

There are $\binom{n}{r}$ ways to pick r girls from town X, and $\binom{n}{r}$ ways to pick r boys from town X, and $r!$ ways to make r pairs of these boys and girls. As all girls in town X know all boys in town X, each girl is then with a boy she knows. Hence

$$X_n(r) = \binom{n}{r}^2 \cdot r! = \frac{(n!)^2}{r!((n-r)!)^2}.$$

Let $A_n(r)$ be the number of different ways in which r girls from town Y can dance with r boys from town Y, forming r pairs, each girl with a boy she knows, such that g_n is one of the girls in the pairs. Let $B_n(r) = Y_n(r) - A_n(r)$.

If girl g_n is not in one of the pairs, then boys b_{2n-2} and b_{2n-1} are not in one of the pairs either. So for $n \geq 2$ and $r \leq n-1$ we have $B_n(r) = Y_{n-1}(r)$.

On the other hand, if girl g_n is in one of the pairs, then we can delete that pair to find a way in which $r-1$ girls from a town with $n-1$ girls can dance with $r-1$ boys from the same town. Given such a set of $r-1$ pairs, we can extend this set to a set of r pairs from a town with n girls by adding the pair (g_n, b_i) for some i. For b_i we can choose from $\{b_1, b_2, \ldots, b_{2n-1}\}$ except that the $r-1$ boys that are already in one of the pairs are not allowed. So there are $(2n-1)-(r-1) = 2n-r$ possibilities for b_i. We conclude that for $n \geq 2$ and $r \geq 2$ we have $A_n(r) = (2n-r)Y_{n-1}(r-1)$.

We will now prove by induction on n that for $r = 1, 2, \ldots, n$ we have

$$Y_n(r) = \frac{(n!)^2}{r!((n-r)!)^2}, \tag{7}$$

which finishes the proof.

For $n = 1$ we just need to prove that $Y_1(1) = 1$. As there is only one girl and one boy, this is trivial. Now let $k \geq 1$ and suppose (7) is true for $n = k$. Then for $r = 2, 3, \ldots, k$ we have

$$\begin{aligned}
Y_{k+1}(r) &= A_{k+1}(r) + B_{k+1}(r) \\
&= (2(k+1)-r)Y_k(r-1) + Y_k(r) \\
&= (2k+2-r)\frac{(k!)^2}{(r-1)!((k-r+1)!)^2} + \frac{(k!)^2}{r!((k-r)!)^2} \\
&= \frac{r(2k+2-r)(k!)^2 + (k-r+1)^2(k!)^2}{r!((k-r+1)!)^2} \\
&= \frac{(k!)^2\big((2kr+2r-r^2) + (k^2+r^2+1-2kr+2k-2r)\big)}{r!((k+1-r)!)^2} \\
&= \frac{(k!)^2(k^2+1+2k)}{r!((k+1-r)!)^2} \\
&= \frac{((k+1)!)^2}{r!((k+1-r)!)^2},
\end{aligned}$$

which is what we wanted to prove. Furthermore, for $r = 1$ we have

$$Y_{k+1}(1) = A_{k+1}(1) + B_{k+1}(1) = (2k+1) + Y_k(1) = (2k+1) + k^2 = (k+1)^2.$$

Finally, for $r = k+1$ we have

$$Y_{k+1}(k+1) = A_{k+1}(k+1) = (k+1)Y_k(k) = (k+1)\frac{(k!)^2}{k!} = \frac{((k+1)!)^2}{(k+1)!}.$$

This completes the induction. $\square$

Problem 4.

We use signed distances throughout the proof. Assume that B is inside segment CE; in the other case a similar proof can be used. We have

$$\angle FAE = \angle DAE = \angle CBF = \angle EBF.$$

As B and F are on different sides of the line AE, we conclude that $ABEF$ is a cyclic quadrilateral. Hence $JA \cdot JB = JE \cdot JF$. Furthermore, K belongs to the circumcircle of ABI if and only if $JA \cdot JB = JK \cdot JI$. Therefore K belongs to the circumcircle of ABI if and only if $JE \cdot JF = JK \cdot JI$. Expressing $JI = JF + FI$, $JE = JF + FE$ and $JK = \frac{1}{2}(JE + JF) = \frac{1}{2}FE + JF$, we find that K belongs to the circumcircle of ABI if and only if

$$\begin{aligned}
(JF + FE)JF &= (\tfrac{1}{2}FE + JF)(JF + FI) && \Leftrightarrow \\
\tfrac{1}{2} \cdot FE \cdot JF &= (\tfrac{1}{2}FE + JF)FI && \Leftrightarrow \\
JF &= \frac{FE \cdot FI}{FE - 2FI}.
\end{aligned}$$

Since $ABEF$ is cyclic and AB is parallel to CD, we have

$$\angle FEC = \angle FEB = 180^\circ - \angle FAB = \angle FDC.$$

Hence $CEDF$ is cyclic as well, yielding $IC \cdot ID = IE \cdot IF$. Furthermore, K belongs to the circumcircle of CDJ if and only if $IC \cdot ID = IK \cdot IJ$. Therefore K belongs to the circumcircle of CDJ if and only if $IE \cdot IF = IK \cdot IJ$. Expressing $IJ = IF + FJ$, $IE = IF + FE$ and $IK = \frac{1}{2}(IE + IF) = \frac{1}{2}FE + IF$, we find that K belongs to the circumcircle of CDJ if and only if

$$\begin{aligned}
(IF + FE)IF &= (\tfrac{1}{2}FE + IF)(IF + FJ) && \Leftrightarrow \\
\tfrac{1}{2} \cdot FE \cdot IF &= (\tfrac{1}{2}FE + IF)FJ && \Leftrightarrow \\
FJ &= \frac{FE \cdot IF}{FE + 2IF} && \Leftrightarrow \\
JF &= \frac{FE \cdot FI}{FE - 2FI}.
\end{aligned}$$

We conclude that K belongs to the circumcircle of ABI if and only if K belongs to the circumcircle of CDJ. $\square$

Comment 1. After deriving that what needs to be proved is: $JE \cdot JF = JK \cdot JI$ if and only if $IE \cdot IF = IK \cdot IJ$, the solution comes down to eliminating all but three well-chosen distances and then manipulating the equalities until it is clear that they are equivalent. The above solution is just one way of doing this.

Comment 2. The conditions $JE \cdot JF = JK \cdot JI$ and $IE \cdot IF = IK \cdot IJ$ are two of the many equivalent ways of expressing that the points E, F and J, I are in harmonic division, i.e. $(EFJI) = -1$, where K is the midpoint of segment EF. Observing this fact would also suffice to finish the solution.

Team Selection Test

Valkenswaard, June 13, 2009

Problem 1. Let $n \geq 10$ be an integer, and consider n in base 10. Let $S(n)$ be the sum of the digits of n. A *stump* of n is a positive integer obtained by removing a number of digits (at least one, but not all) from the right side of n. E.g.: 23 is a stump of 2351. Let $T(n)$ be the sum of all the stumps of n.
Prove that $n = S(n) + 9 \cdot T(n)$.

Problem 2. Let ABC be a triangle, P the midpoint of BC, and Q a point on segment CA such that $|CQ| = 2|QA|$. Let S be the intersection of BQ and AP. Prove that $|AS| = |SP|$.

Problem 3. Let a, b and c be positive reals such that $a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}\, abc.$$

Problem 4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

for all $m, n \in \mathbb{Z}$.

Problem 5. Suppose that we are given an n-gon of which all sides have the same length, and of which all the vertices have rational coordinates. Prove that n is even.

Solutions of the Team Selection Test 2009

Problem 1.

Let us denote the digits of n from right to left by a_0, a_1, $\ldots$, a_k. We have

$$n = a_0 + 10a_1 + \cdots + 10^k a_k.$$

A stump of n consists of (from right to left) the digits a_i, a_{i+1}, $\ldots$, a_k, where $1 \le i \le k$. We see that such a stump is equal to $a_i + 10a_{i+1} + \cdots + 10^{k-i}a_k$. Summation over i then yields $T(n)$. Now write $T(n)$ in a different way, by taking all the terms involving a_1, then those involving a_2, and so on (effectively, as we will see below, we are changing the order of summation; it does not matter whether we first sum over i from 1 up to k, and then, for each i, sum over j from i up to k, or we first sum over the j from 1 up to k, and then, for each j, sum over i from j up to k):

$$T(n) = \sum_{i=1}^{k} \left(a_i + 10a_{i+1} + \cdots + 10^{k-i}a_k\right) = \sum_{i=1}^{k}\sum_{j=i}^{k} 10^{j-i}a_j$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{j} 10^{j-i}a_j = \sum_{j=1}^{k} \left(1 + 10 + \cdots + 10^{j-1}\right) a_j = \sum_{j=1}^{k} \frac{10^j - 1}{10 - 1} a_j,$$

where we used the summation formula for the geometric series in the final step. So we get

$$9 \cdot T(n) = \sum_{j=1}^{k} (10^j - 1)a_j = \sum_{j=0}^{k} (10^j - 1)a_j.$$

Recall that $S(n) = \sum_{j=0}^{k} a_j$. Hence we have

$$S(n) + 9 \cdot T(n) = \sum_{j=0}^{k} (10^j - 1 + 1)a_j = \sum_{j=0}^{k} 10^j a_j = n.$$

$\square$

Problem 2.

Solution 1. Let T be a point on BQ such that PT is parallel to AC. Then PT joins the midpoints of BC and BQ, so $|PT| = \frac{1}{2}|CQ| = |QA|$. So we see that $ATPQ$ is a quadrilateral of which two sides are parallel and of the same length. This implies that it is a parallelogram. Since any parallelogram has the property that its diagonals bisect each other, we see that $|AS| = |SP|$. $\square$

Solution 2. We apply Menelaos' Theorem to triangle PCA. Since the points B, S and Q are collinear, we have

$$-1 = \frac{PB}{BC} \cdot \frac{CQ}{QA} \cdot \frac{AS}{SP} = \frac{-1}{2} \cdot \frac{2}{1} \cdot \frac{AS}{SP} = -\frac{AS}{SP}.$$

Hence $\frac{AS}{SP} = 1$, from which we can deduce that S is the midpoint of the segment AP. □

Solution 3. Let M be the midpoint of QC and let $x = [AQS] = [QMS] = [MCS]$ and $y = [CPS] = [PBS]$. Since $[CPA] = [PBA]$, we see that $[ASB] = 3x$. But then we have $[AQB] = x + 3x$, whereas on the other hand, we have $2[AQB] = [QCB]$, so $2x + 2y = [QCB] = 2[AQB] = 8x$. Hence $y = 3x$. But then $[ASB] = 3x = y = [SPB]$, from which we conclude that $|AS| = |SP|$. □

Solution 4. Let R be the intersection of CS and AB. According to Ceva's Theorem, we have

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1,$$

from which we deduce that $2|AR| = |RB|$. Let $a = [PSC]$, $b = [QSC]$, $c = [QSA]$, $d = [RSA]$, $e = [RSB]$ and $f = [PSB]$. Then we see that

$$b + a + f = 2(c + d + e) \quad \text{en} \quad a + e + f = 2(b + c + d),$$

which implies that

$$b - e = 2e - 2b,$$

hence $b = e$. Furthermore, we have $2d = e$ and $2c = b$, hence $c = d$ and $c + d = e$. Now $a + e + f = 2(b + c + d)$ implies that

$$a + f \;=\; b + 2c + 2d \;=\; b + c + d + e,$$

so

$$2(a + f) \;=\; a + b + c + d + e + f \;=\; [ABC].$$

Hence $2|PS| = |PA|$, implying that $|PS| = |AS|$. □

Problem 3.

First of all, note that $a^2 + b^2 + c^2 \geq ab + bc + ca$, and hence that $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$. Applying AM-GM we see that $a + b + c \geq 3(abc)^{\frac{1}{3}}$. On the other hand, note that $a + b + c \geq abc$. Now we have the following two inequalities:

$$a^2 + b^2 + c^2 \geq \tfrac{1}{3}(a + b + c)^2 \geq 3(abc)^{\frac{2}{3}},$$
$$a^2 + b^2 + c^2 \geq \tfrac{1}{3}(a + b + c)^2 \geq \tfrac{1}{3}(abc)^2.$$

Raising the former equation to the $\frac{3}{4}$-th power, and the latter one to the $\frac{1}{4}$-th (which is allowed, since every expression is positive):

$$(a^2 + b^2 + c^2)^{\frac{3}{4}} \geq 3^{\frac{3}{4}}(abc)^{\frac{1}{2}},$$
$$(a^2 + b^2 + c^2)^{\frac{1}{4}} \geq 3^{-\frac{1}{4}}(abc)^{\frac{1}{2}}.$$

Taking the product of these two equations, we then see that

$$a^2 + b^2 + c^2 \geq 3^{\frac{1}{2}}(abc),$$

which is as desired. □

Problem 4.

Suppose that there exists a $c \in \mathbb{Z}$ such that $f(n) = c$ for all n. Then $2c = c^2 + 2$, so we get the equation $c^2 - 2c + 2 = 0$, which has no solutions in $\mathbb{Z}$. Hence f cannot be constant. Now substitute $m = 0$. This yields the equation $f(n) + f(-1) = f(n)f(0) + 2$, from which we deduce that $f(n)(1 - f(0))$ is constant. Since $f(n)$ isn't a constant function, we see that $f(0) = 1$. Using the same equation, we then get $f(-1) = 2$. Now substitute $m = n = -1$. This yields $f(-2) + f(0) = f(-1)^2 + 2$, from which follows that $f(-2) = 5$. Substituting $m = 1$ and $n = -1$ now yields $f(0) + f(-2) = f(1)f(-1) + 2$, which implies that $f(1) = 2$.

Now substitute $m = 1$, then we obtain $f(n+1) + f(n-1) = f(1)f(n) + 2$, or, equivalently

$$f(n+1) = 2f(n) + 2 - f(n-1).$$

By induction, it then follows that $f(n) = n^2+1$ for all non-negative n, and, using the equation

$$f(n-1) = 2f(n) + 2 - f(n+1)$$

also for all non-positive n. Hence $f(n) = n^2 + 1$ for all n, and we can easily check that this function satisfies the given equation. $\square$

Problem 5.

Let (x_1, y_1), $\ldots$, (x_n, y_n) be the coordinates of the vertices of the given n-gon. Define $a_i = x_{i+1} - x_i$, $b_i = y_{i+1} - y_i$ for $i = 1, 2, \ldots, n$, where $x_{n+1} = x_1$ and $y_{n+1} = y_1$. Then we note that $a_i, b_i \in \mathbb{Q}$ and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$; and that the sum $a_i^2 + b_i^2$ does not depend on i. By multiplying with a suitable factor we can get rid of denominators and common divisors of the a_i and b_i, so we may assume that $a_i, b_i \in \mathbb{Z}$ and that $\gcd(a_1, \ldots, a_n, b_1, \ldots, b_n) = 1$. Let c be the integer such that $a_i^2 + b_i^2 = c$ for all i.

Suppose that c is odd. Then for every i, exactly one of a_i, b_i is odd. Hence of the $2n$ integers a_i, b_i, exactly n are odd. Then we get $0 = \sum_{i=1}^n (a_i + b_i) \equiv n \mod 2$, implying that n is even.

Now suppose that c is even. Then, for all i, we have $a_i \equiv b_i \mod 2$. If there exists an i such that a_i and b_i are both odd, then $c = a_i^2 + b_i^2 \equiv 1 + 1 \equiv 2 \mod 4$. If there exists an i such that a_i and b_i are both even, then $c = a_i^2 + b_i^2 \equiv 0 + 0 \equiv 0 \mod 4$. These two cases cannot occur together. Hence either all of the a_i and b_i are odd, or all of them are even. The latter statement contradicts our assumption about the greatest common divisor of the a_i and b_i. The former statement yields $0 = \sum_{i=1}^n a_i \equiv n \mod 2$, implying again that n is even. $\square$

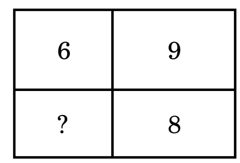
Junior Mathematical Olympiad

October 3, 2008
Vrije Universiteit Amsterdam

Problems part 1

- The problems in part 1 are five-choice questions. At each question exactly one of the given five answers is correct. Indicate clearly on the answer sheet which letter corresponds to the right answer.
- Each correctly given answer will get you 2 points. For wrong answers no points are subtracted.
- You are allowed scrap paper, as well as a compass and a ruler or protractor. Calculators and other electronic devices are not allowed.
- The time allowed for this part is 60 minutes.
- Good luck!

1. A rectangle has been divided into four smaller rectangles. The areas of three of the small rectangles are 6, 8 and 9 (see the figure). Determine the area of the fourth small rectangle.



A) $4\frac{2}{3}$ B) 5 C) $5\frac{1}{3}$ D) $5\frac{2}{3}$ E) 7

2. We multiply all odd numbers between 0 and 100. What is the last digit of the result?

A) 1 B) 3 C) 5 D) 7 E) 9

3. Six students are sitting next to each other on six chairs, numbered from 1 to 6. Now they all get up at the same time, and then they sit down on a chair again, according to the table below.

standing up from chair number	1	2	3	4	5	6
	↓	↓	↓	↓	↓	↓
sitting down on chair number	4	3	1	6	5	2

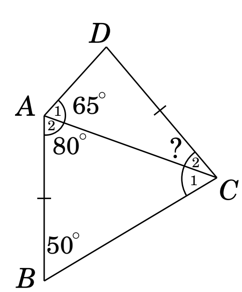
This process of standing up and sitting down again happens 642 times. Which chair is now occupied by the student who was at the start sitting on chair number 1?

A) 1 B) 2 C) 3 D) 4 E) 6

4. Consider the sequence of positive integers $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$ in which the n-th positive integer occurs exactly n times. We divide the hundredth number in the sequence by 5. What is the remainder?

A) 0 B) 1 C) 2 D) 3 E) 4

5. In a quadrilateral $ABCD$ the sides AB and CD have equal length. Moreover, three angles are given: $\angle A_1 = 65°$, $\angle A_2 = 80°$ and $\angle B = 50°$. Determine $\angle C_2$.



A) 30° B) 40° C) 50° D) 60° E) 65°

6. How many (positive or negative) integers n exist such that $\frac{12}{n+5}$ is an integer?

A) 2 B) 6 C) 8 D) 10 E) 12

7. What is

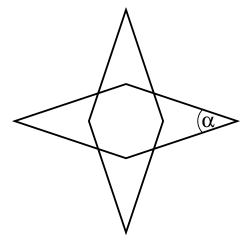
$$\left(1-\frac{1}{4}\right)\times\left(1-\frac{1}{9}\right)\times\left(1-\frac{1}{16}\right)\times\left(1-\frac{1}{25}\right)\times\left(1-\frac{1}{36}\right)\times\left(1-\frac{1}{49}\right)?$$

A) $\frac{1}{7}$ B) $\frac{2}{7}$ C) $\frac{3}{7}$ D) $\frac{4}{7}$ E) $\frac{5}{7}$

8. How many integers of the form $bbcac$ exist, where a, b and c are digits $(0, 1, \ldots, 9)$ with $c > a$ and with b equal to the mean of a and c?

A) 20 B) 21 C) 22 D) 23 E) 24

9. Two identical rhombuses are lying on top of each other, one of them rotated 90 degrees compared to the other. The area where the two rhombuses overlap, happens to be a regular octagon: all eight sides have equal length and all eight angles have equal sizes. Determine the smallest angle of the rhombus, indicated in the figure by α.



A) 22,5° B) 30° C) 40° D) 45° E) 60°

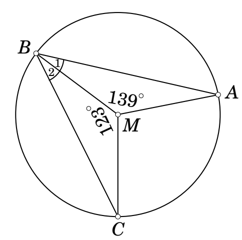
10. How many of the integers $1, 2, 3, \ldots, 2008$ are *not* divisible by 2 and *not* divisible by 5 either?

A) 403 B) 603 C) 803 D) 1205 E) 1405

11. A large pond is being emptied by means of three pumps. Using just the first pump, emptying the pond would take four days. Using just the second pump, it would take three days, and using just the third pump, it would take two days. How many days does it take to empty the pond using all three pumps at the same time?

A) $\frac{1}{9}$ day B) $\frac{12}{13}$ day C) 1 day D) $\frac{13}{12}$ day E) 3 days

12. Through the vertices A, B and C of a triangle ABC passes a circle with midpoint M. Of the three angles at the midpoint, two are equal to 123° and 139° (see figure). Determine $\angle B_{12}$.



A) 45° B) 49° C) 50° D) 51° E) 59°

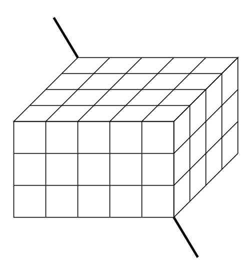
13. If we divide the number $2^{2008} - 2^{2007} + 2^{2006} - 2^{2005} + 2^{2004} - 2^{2003} + 2^{2002} - 2^{2001}$ by the number 2^{2000}, then the result is an integer (that is, there is no remainder). Determine this integer.

A) 4 B) 36 C) 72 D) 170 E) 200

14. Consider an integer $abcd$ consisting of four distinct digits a, b, c and d (where a is not allowed to be 0). Of this integer you know that $abcd \times 11 = ac9bd$. The digit b can be equal to only one of the following five options. Which one?

A) 2 B) 3 C) 4 D) 5 E) 6

15. A block of cheese consisting of $3 \times 4 \times 5$ small cubes of cheese, has been perforated by a thin pin along a space diagonal. How many of the small cubes have been perforated by the pin?



A) 5 B) 8 C) 10 D) 11 E) 12

Junior Mathematical Olympiad

October 3, 2008
Vrije Universiteit Amsterdam

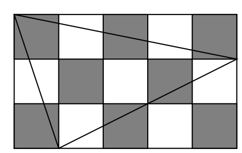
Problems part 2

- The problems in part 2 are open questions. At each question your answer should be a number or an expression (for example $2\frac{2}{3}$ or a^2). Write this answer on your answer sheet at the indicated place.
- Each correctly given answer will get you 2 points. For wrong answers no points are subtracted.
- You are allowed scrap paper, as well as a compass and a ruler or a protractor. Calculators and other electronic devices are not allowed.
- The time allowed for this part is 60 minutes.
- Good luck!

1. What is the maximum number of points of intersection between a circle and a triangle?

2. During a tournament with six players, each player plays a match against each other player. At each match there is a winner; ties do not occur. A journalist asks five of the six players how many matches each of them has won. The answers given are 4, 3, 2, 2 and 2. How many matches have been won by the sixth player?

3. A 3 by 5 rectangle has been coloured like a chess board. What is the total area of the black parts inside the triangle drawn in the figure?

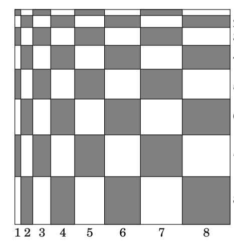


4. I have written four numbers on a piece of paper. In six different ways I can choose two and add them up. The resulting sums are 11, 15, 16, 16, 17 and 21. Now I multiply all of the four numbers on my piece of paper. What is the result?

5. What is the sum of the positive integers smaller than 100,000 (a hundred thousand) in which only the digits 0 and 1 occur?

6. Find the smallest positive integer with the property that if you multiply its digits, the result is 1890.

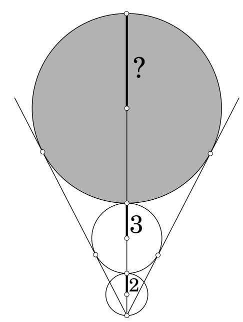
7. In this stretched chess board the small rectangles have sides with length 1 up to 8. Determine the total area of all black rectangles.



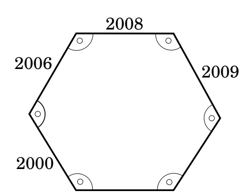
8. Write the following expression as one fraction in its simplest terms, given that for a, b and c we have $a \times b \times c = 1$.

$$\frac{1}{1 + a + (a \times b)} + \frac{1}{1 + b + (b \times c)} + \frac{1}{1 + c + (c \times a)}$$

9. The two smallest circles in the figure have radiuses 2 and 3. Find the radius of the largest circle.



10. A hexagon has six angles all equal to 120 degrees. The lengths of four consecutive sides are 2000, 2006, 2008 and 2009. Determine the perimeter of the hexagon.



First Round
Dutch Mathematical Olympiad

Friday, January 30, 2009

Problems

- Time available: 2 hours.
- The A-problems are multiple choice questions. Only one of the five options given is correct. Please state clearly which letter precedes the correct solution. Each correct answer is worth 2 points.
- The B-problems are open questions; the answers to these questions are a number, or numbers. Each correct answer is worth 5 points. Please work accurately, since an error in your calculations may cause your solution to be considered incorrect and then you won't get points at all for that question. Please give your answers exactly, for example $\frac{11}{81}$ or $2 + \frac{1}{2}\sqrt{5}$ or $\frac{1}{4}\pi + 1$.
- You are not allowed to use calculators and formula sheets; you can only use a pen, a compass and a ruler or set square. And your head, of course.
- This is a competition, not an exam. The main thing is that you have fun solving unusual mathematical problems. Good luck!

A-problems

A1. Ella has answered three sets of questions. Of the first set, consisting of 25 questions, she answered 60% correctly. Of the second set, consisting of 30 questions, she answered 70% correctly, and of the third set, consisting of 45 questions, she answered 80% correctly. Now, if we combine the three sets to form one set of 100 questions, what percentage of these 100 questions did Ella answer correctly?

(A) 68% (B) 70% (C) 72% (D) 74% (E) 76%

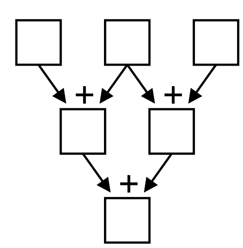
A2. How many of the integers from 10 to 99 (10 and 99 included) have the property that the sum of their digits is equal to the square of an integer? (An example: The sum of the digits of 27 is equal to $2 + 7 = 9 = 3^2$.)

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

A3. Ronald rolls three dice. These dice look like normal dice, but the numbers on their sides are unusual.
On the sides of the first dice are the numbers 1, 1, 2, 2, 3, 3.
On the sides of the second dice are the numbers 2, 2, 4, 4, 6, 6.
On the sides of the third dice are the numbers 1, 1, 3, 3, 5, 5.
He then adds up the three numbers he gets from rolling the three dice.
What is the probability that the resulting number is odd?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

A4. Three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are placed in the three squares at the top of the figure to the right, after which the numbers are added as indicated in said figure. We call Max the highest number that can appear in the bottom square, and Min the lowest number that can appear there. What is the value of Max – Min?



(A) 16 (B) 24 (C) 25 (D) 26 (E) 32

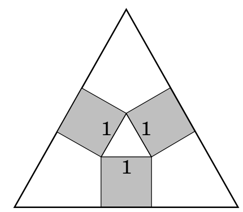
A5. The ratio between the lengths of the diagonals of a rhombus is 3 to 4. (A rhombus is an equilateral quadrilateral.) The sum of the lengths of the diagonals is 56. What is the perimeter of this rhombus?

(A) 80 (B) 96 (C) 100 (D) 108 (E) 160

A6. Wouter walks from his home to the fitness center. He could also have chosen to go by bike, in which case he would have covered the distance between his home and the fitness center 7 times as fast. However, he has left his bike at home. After having walked 1 km he reaches a bridge. Continuing on foot will take Wouter just as long as walking back home to get his bike and then cycle to the fitness center. What is the distance between the bridge and the fitness center in kilometers?

(A) $\frac{8}{7}$ (B) $\frac{7}{6}$ (C) $\frac{6}{5}$ (D) $\frac{5}{4}$ (E) $\frac{4}{3}$

A7. On the sides of an equilateral triangle, we draw three squares. The sides of these squares that are parallel to the sides of the triangle are extended until they intersect. These three intersections form another equilateral triangle. Suppose that the length of a side of the original triangle is equal to 1. What is the length of a side of the large equilateral triangle?



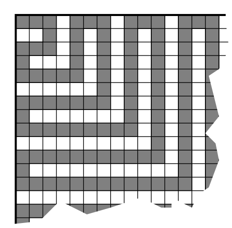
(A) $1 + 2\sqrt{2}$ (B) $5 - \frac{1}{2}\sqrt{3}$ (C) $3\sqrt{2}$ (D) $1 + 2\sqrt{3}$ (E) $2\sqrt{6}$

A8. Consider all four-digit numbers in which each of the digits 3, 4, 6 and 7 occurs exactly once. How many of these numbers are divisible by 44?

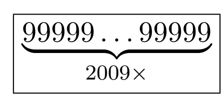
(A) 2 (B) 4 (C) 6 (D) 8 (E) 12

B-problems

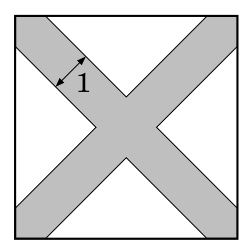
B1. A sheet of paper shows a grid of 101 by 101 white squares. A chain is formed by coloring squares grey as shown in the figure to the right. The chain starts in the upper left-hand corner and goes on until it cannot go on any further. Only part of the grid is shown. In total, how many squares are colored grey in the original grid of 101 by 101 squares?



B2. The integer N consists of 2009 consecutive nines. A computer calculates $N^3 = (99999\ldots99999)^3$. How many nines does the number N^3 contain in total?



B3. Using a wide brush, we paint the diagonals of a square tile, as in the figure. Exactly half of the surface of this tile is covered with paint. Given that the width of the brush is 1, as indicated in the figure, what is the length of the side of the tile?



B4. Determine a triplet of integers (a, b, c) satisfying:

$$a + b + c = 18$$
$$a^2 + b^2 + c^2 = 756$$
$$a^2 = bc$$

First Round
Dutch Mathematical Olympiad

Friday, January 30, 2009

Solutions

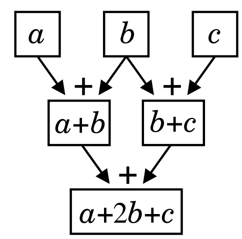
A1. (C) 72% 60% of 25 is 15; 70% of 30 is 21; and 80% of 45 is 36. So in total, Ella answered $15 + 21 + 36 = 72$ of the 100 questions correctly.

A2. (E) 17 We check how many of these numbers have sum of digits equal to 1, 2, etc. There is 1 number with sum 1 (being 10); there are 2 with sum 2 (being 20 and 11); etc.; 9 with sum 9 (being $90, 81, \ldots, 18$); also, 9 with sum 10 (being $91, 82, \ldots, 19$); etc.; and finally, 1 with sum 18 (being 99); see the table below. Then the sum of digits is a square of an integer (i.e. 1, 4, 9 or 16) in $1 + 4 + 9 + 3 = 17$ of the 90 cases.

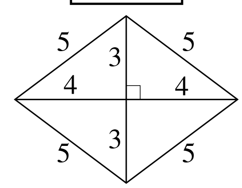
sum:	**1**	2	3	**4**	5	6	7	8	**9**	10	11	12	13	14	15	**16**	17	18
number:	**1**	2	3	**4**	5	6	7	8	**9**	9	8	7	6	5	4	**3**	2	1

A3. (B) $\frac{1}{3}$ Note that the second die only has even numbers on it, and that the third die only has odd numbers on it. So essentially the question is to find the probability that rolling the first die gives an *even* number. Since 2 of the 6 numbers on this die are even, this probability is equal to $\frac{2}{6} = \frac{1}{3}$.

A4. (D) 26 Let's say we put a, b, and c in the top three squares. Then the result in the bottom square is $a + 2b + c$. So we can maximize this result by making first b, then a and c as large as possible. Taking $b = 9, a = 8$ and $c = 7$ then yields 33 as result. In the same way, we can minimize the result by making first b, then a and c as small as possible. Taking $b = 1$, $a = 2$, $c = 3$ yields 7 as result. The difference between these numbers is $33 - 7 = 26$.

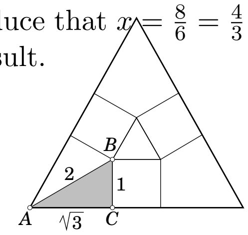


A5. (A) 80 Note that the diagonals have lengths $\frac{3}{7} \cdot 56 = 24$ and $\frac{4}{7} \cdot 56 = 32$. So the halves of diagonals have lengths $12 = 3 \cdot 4$ and $16 = 4 \cdot 4$. So the rhombus is 4 times larger than the rhombus in the figure, which consists of four triangles with sides 3, 4 and 5. Hence the sides of the original rhombus have length $4 \cdot 5 = 20$, and thus the perimeter has length $4 \cdot 20 = 80$.



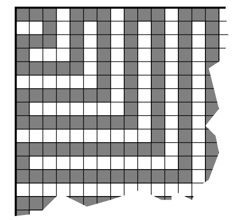
A6. (E) $\frac{4}{3}$ Let x be said distance and let us suppose that he has spent a quarter of an hour walking by then. Then continuing walking will take him x quarters of an hour. On the other hand, if he decides to walk back home to pick up his bike, he'll first have to spend one quarter of an hour to get back, and then $\frac{1+x}{7}$ quarters of an hour by bike; since he travels 7 times faster that way. Then we have $x = 1 + \frac{1+x}{7}$, so $7x = 7 + (1 + x)$, or $6x = 8$. We deduce that $x = \frac{8}{6} = \frac{4}{3}$. Taking for 'quarter of an hour' any other time unit will give us the same result.

A7. (D) $1 + 2\sqrt{3}$ In $\triangle ABC$, $\angle A$ is half of 60°, so 30°. Also, $\angle C$ is a right angle, so $\triangle ABC$ is a 30°-60°-90°-triangle, where $|BC| = 1$. So it's half of an equilateral triangle with sides 2: $|AB| = 2$. Now we calculate $|AC|$ with the Theorem of Pythagoras: $|AC| = \sqrt{2^2 - 1^2} = \sqrt{3}$. So the required length is $\sqrt{3} + 1 + \sqrt{3}$.



A8. (A) 2 Suppose that n, having digits a, b, c and d (so $n = 1000a + 100b + 10c + d$) is divisible by 44. Then it is also divisible by 11. Since the number $m = 1001a + 99b + 11c$ is also divisible by 11, so is $m - n$. Hence $m - n = a - b + c - d$ is a multiple of 11. But this number is at most the sum of the two highest digits, minus the sum of the lowest two, so $13 - 7 = 6$, and in the same way, we see that this number is at least -6. So it has to be equal to 0. Thus $a + c = b + d$, and since the sum of the digits is 20, we have $a + c = b + d = 10$. First suppose that $d = 4$. Then $b = 6$ so we get two possibilities for n, namely 3674 and 7634. But neither of these is divisible by 4, let alone by 44. Now suppose that $d = 6$, then we have $b = 4$, and in this case we get 3476 and 7436, both of which are divisible by 44. Finally, note that since n is divisible by 4, d must be even. We deduce that we have only 2 such numbers that are divisible by 44.

B1. 5201 We can subdivide this grid of 101^2 squares as follows. In the upper left corner, we have one (grey) square, then two L-shaped pieces, one having 3 squares (one of which grey), the other having 5 squares (all of which grey). Then we have two more L-shaped pieces, one having 7 squares (one of which grey), the other having 9 squares (all of which grey), etc. Of the last two L-shaped pieces, the first one has 199 squares (one of which grey), and the second one has 201 squares (all of which grey). We have 50 pairs of L-shapes in total, so the total number of grey squares is $1+(1+5)+(1+9)+(1+13)+\ldots+(1+201) = 1+(6+10+14+\ldots+202) = 1+\frac{1}{2}\cdot 50\cdot(6+202) = 5201$.

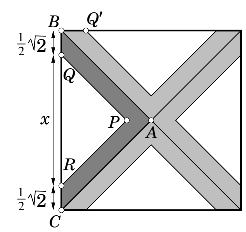


Alternative solution In each pair of these L-shapes, there are 4 more grey squares than white squares. So there are $50\cdot 4 = 200$ more grey squares than white squares in the $101^2-1 = 10200$ squares contained in the 50 pairs of L-shapes, so we have 5000 white squares and 5200 grey ones. Since the upper left square is grey, in total, we have 5201 grey squares.

B2. 4017 $9^3 = 729$; $99^3 = 970299$; $999^3 = 997002999$. It seems to be the case that in general, the third power of a number n consisting of k consecutive nines takes the following form: first $k-1$ nines; then a 7; then $k-1$ zeroes; then a 2; and finally k nines. To prove this, we write $n = 10^k-1$. Indeed: $(10^k-1)^3 = 10^{3k}-3\cdot 10^{2k}+3\cdot 10^k-1 = 10^{2k}(10^k-3)+(3\cdot 10^k-1)$. The number 10^k-3 can be written as $999\ldots997$ with $k-1$ nines. Multiplied with 10^{2k} this gives a number that ends in $2k$ zeroes. Adding $3\cdot 10^k-1$ to this number, the last $k+1$ zeroes are replaced with $2999\ldots999$ with k nines. So in total, we have $(k-1)+k$ nines; in our case, $k = 2009$, so we have 4017 nines.

$$\boxed{\underbrace{99999\ldots99999}_{2009\times}}$$

B3. $2+2\sqrt{2}$ We only need to look at a quarter of the tile: $\triangle ABC$. The area of $\triangle PQR$ is half of the area of $\triangle ABC$. The triangles are similar, so corresponding sides have a ratio of $1:\sqrt{2}$, so $|QR|:|BC| = 1:\sqrt{2}$.
Now we calculate $|BQ|$ using the Theorem of Pythagoras in $\triangle BQQ'$: $2|BQ|^2 = |BQ'|^2+|BQ|^2 = 1^2$, so $|BQ| = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$. Now let us write x for $|QR|$. Then we find $x+\sqrt{2} = \sqrt{2}\cdot x$, so $x(\sqrt{2}-1) = \sqrt{2}$ or equivalently, $x = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = 2+\sqrt{2}$. Hence $|BC| = x+\sqrt{2} = 2+2\sqrt{2}$ (or $|BC| = \sqrt{2}\cdot x = \sqrt{2}\cdot(2+\sqrt{2}) = 2\sqrt{2}+2$).



B4. $(a,b,c) = (-12,6,24)$ of $(a,b,c) = (-12,24,6)$ (one answer is enough)
We calculate $(b+c)^2$ in two different ways. $(b+c)^2 = (18-a)^2 = 324-36a+a^2$ and $(b+c)^2 = b^2+2bc+c^2 = (756-a^2)+2a^2$. So $a^2-36a+324 = a^2+756$, or $-36a = 756-324 = 432$, so $a = -12$. Substituting this in the first and in the last equation, we obtain the equations $b+c = 30$ and $bc = 144$. Trying some divisors of $144 = 12^2$, we then should be able to find a solution. Or we can just substitute $c = 30-b$ in the last equation, yielding the quadratic equation $b(30-b) = 144$, or equivalently $b^2-30b+144 = 0$. We can factorize this as $(b-6)(b-24) = 0$ (or we can use the abc-formula) to see that we have two solutions $b = 6$ (and $c = 24$) or $b = 24$ (and $c = 6$).
Alternative solution Just as above, we see that $a = -12$. Then by substituting this in all three equations, we see that $b+c = 30$, $b^2+c^2 = 756-144 = 612$ and $bc = 144$. Combining the last two equations yields $(b-c)^2 = b^2+c^2-2bc = 612-2\cdot 144 = 324$, or equivalently $b-c = \pm\sqrt{324} = \pm 18$. Adding the equations $b+c = 30$ and $b-c = -18$, we get $2b = (b+c)+(b-c) = 12$ so $b = 6$ (and $c = 24$). Adding the equations $b+c = 30$ and $b-c = 18$, we get $2b = (b+c)+(b-c) = 48$ so $b = 24$ (and $c = 6$).