The 46th

Dutch

Mathematical

Olympiad

2007

and the team selection

for IMO 2008 Madrid

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Contents

Introduction

The Dutch Mathematical Olympiad consists of two rounds. The first round is held on the participating schools and consists of eight multiple choice questions and four open questions (see page 1–4). Students get two hours to work on the paper. On January 26, 2007, in total 2742 students of 185 secondary schools participated in this first round.

Those students that scored 20 points or more on the first round (out of a maximum of 36 points) were invited to the second round, which took place at University of Technology Eindhoven after the summer holidays. Also some outstanding participants in the Kangaroo math contest or the Pythagoras Olympiad were invited.

In total 132 students were invited, of which 119 participated in the second round on September 14th, 2007. The second or final round contains five problems for which the students have to give extensive solutions and proofs. They have three hours for the paper (see page 5–7). After the prices had been awarded in the beginning of November, the Dutch Mathematical Olympiad concluded its 46th edition 2007. In 2011 we will have our 50th edition.

The 24 most outstanding candidates of the Dutch Mathematical Olympiad 2007 were invited to an intensive seven-month training programme, consisting of weekly problem sets. Also, the students met twice for a three-day training camp, three times for a day at the university, and finally for a six-day training camp in the beginning of June. At this time, only 14 candidates were left. Out of those, the team was selected by a final selection test on June 7, 2008 (see page 8–12). The team will have a training camp in Spain from July 7 until July 14, together with the team from New Zealand.

In the meantime, a new edition of the Dutch Mathematical Olympiad has started at the participating schools on January 25, 2008 (see page 13–16). We are happy to see that the number of participating schools as well as the number of participants has increased to 201 respectively 3004. As a new initiative, we have made some training material and we have organized training sessions at six universities in the country for the 144 students who have been invited for the second round in September 2008.

The Dutch team for IMO 2008 Madrid consists of

- Raymond van Bommel (16 y.o., participated in IMO 2007 as well)
- Remy van Dobben de Bruyn (17 y.o.)
- Floris van Doorn (17 y.o., observer C in IMO 2007)
- Alexander van Hoorn (18 y.o.)
- Milan Lopuhaä (18 y.o., participated to in IMO 2007 as well)
- Maarten Roelofsma (17 y.o.)

The Dutch delegation for IMO 2008 Madrid further consists of

- dr. Quintijn Puite, team leader
  University of Technology Eindhoven
- Birgit van Dalen MSc, deputy team leader
  Leiden University
- drs. Wim Berkelmans, project manager IMO 2011, observer A
  VU University Amsterdam
- prof.dr. Gerhard Wöginger, observer A
  University of Technology Eindhoven
- Teunis van Wijngaarden, observer C
  CWI Amsterdam

We are grateful to Jinbi Jin for the translation into English of both the problems and the solutions.

Dutch Mathematical Olympiad
Round 1
Friday, January 26, 2007
Available time: 2 hours

Please read the following before you start:

- The A-problems are multiple choice questions. Only one of the five given options is correct. Please state clearly behind which letter the correct solution is stated. You get 2 points for each correct answer.
- The B-problems are open questions, which have a number as answer. You get 5 points for each correct answer. Please work accurately, since an error in your calculations can cause your solution to be considered wrong. Please give exact answers, for example $\frac{11}{81}$ or $2+\sqrt{3}$ or $\pi+1$.
- This is a competition, not an exam. So it is to be expected that only few people will get all the answers right, and you don't need to worry if you have only solved a part of the problems.
- The use of formula sheets, calculators and similar tools is **not** allowed.
- The point is that you have fun solving unusual mathematical problems. *Good luck!*

A-problems

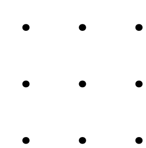
A1. The number M has 2007 digits, each of which is equal to 1, $M = 111\ldots111$. What is the sum of the digits of $2007 \times M$?

A) 2007 B) 18036 C) 18063 D) 18084 E) 4028049

A2. The next five sequences all contain the same numbers. Which sequence is ordered correctly?

A) $0.16 < \frac{1}{7} < \frac{13}{97} < \frac{17}{101} < \frac{5}{33}$
B) $\frac{17}{101} < 0.16 < \frac{1}{7} < \frac{5}{33} < \frac{13}{97}$
C) $\frac{1}{7} < 0.16 < \frac{5}{33} < \frac{17}{101} < \frac{13}{97}$
D) $\frac{5}{33} < \frac{1}{7} < \frac{13}{97} < \frac{17}{101} < 0.16$
E) $\frac{13}{97} < \frac{1}{7} < \frac{5}{33} < 0.16 < \frac{17}{101}$

A3. In the figure, nine grid points are drawn. In how many ways can we draw a triangle, the vertices of which lie on three of these nine grid points?

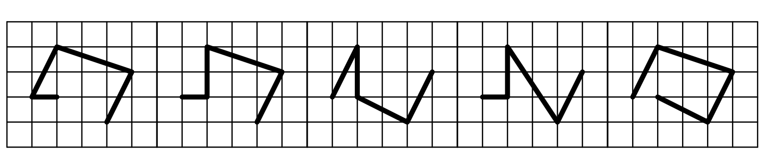


A) 76 B) 84 C) 92 D) 496 E) 504

A4. How many pairs of positive integers (a, b) with $a + b < 100$ satisfy the equation $a + \frac{1}{b} = 13 \times (b + \frac{1}{a})$?

A) 5 B) 7 C) 9 D) 13 E) 28

A5. We use five different paths to connect five grid points.
Which of these paths is the shortest one?



A) B) C) D) E)

A6. We construct a sequence of numbers as following:
The first number is equal to 2, and so is the second number. Every following number is the product of the two previous ones. So the first few numbers of the sequence are: $2, 2, 4, 8, 32, \ldots$
What is the final digit of the 2007th number in this sequence?

A) 0 B) 2 C) 4 D) 6 E) 8

A7. Does the equation $9^n + 9^n + 9^n = 3^{2007}$ have any integral solutions? If so, which of the following is a solution?

A) 667 B) 669 C) 1003 D) 2006 E) No solutions exist

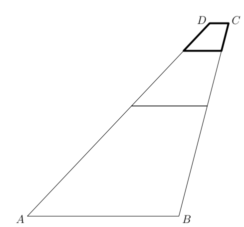
A8. Behind a long table, nine chairs are arranged in a row, on which six students and three teachers are going to sit. The three teachers arrive first and they decide to pick their chairs in such a way that every teacher sits between two students. In how many ways can the teachers pick their chairs, given this constraint?

A) 12 B) 36 C) 60 D) 84 E) 630

B-problems

B1. Mark one card with a '1', two cards with a '2', ..., fifty cards with a '50'. Put these $1 + 2 + \cdots + 50 = 1275$ cards into a box and shuffle them. How many cards do you need to take from the box to be certain that you will have taken at least 10 cards with the same mark?

B2. We have a quadrilateral $ABCD$ with side lengths $|AB| = 16$, $|BC| = 21$, $|CD| = 2$ and $|DA| = 28$. The sides AB and CD are parallel. Two lines, both parallel to AB and CD, divide $ABCD$ into three similar quadrilaterals.
Calculate the perimeter of the smallest of these quadrilaterals.

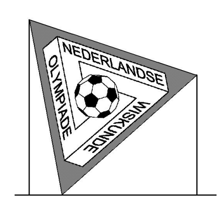


B3. For every pair of integers a, b, we define an operator $a \otimes b$ with the following three properties.

1. $a \otimes a = a + 2$;
2. $a \otimes b = b \otimes a$;
3. $\frac{a \otimes (a+b)}{a \otimes b} = \frac{a+b}{b}$.

Calculate $8 \otimes 5$.

B4. A flag with the shape of an equilateral triangle is hung up on the tops of two vertical poles by two of its vertices. One of these poles has length 4, while the other has length 3. Also, the third vertex of the flag touches the ground. Calculate the length of the flag's side.



Solutions
of the problems of Round 1 of the
Dutch Mathematical Olympiad 2007

A-problems

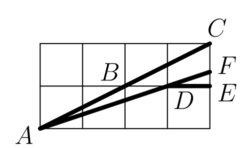
A1. C) 18063 If M had 4 ones, then $2007 \times M = 2229777$, and the sum of the digits equals $3 \times (2+7) + 9 = 4 \times 9$.
If M had 5 ones, then $2007 \times M = 22299777$, and the sum of the digits equals $3 \times (2+7) + 2 \times 9 = 5 \times 9$.
Continuing like this, we see that if M has 2007 ones, then the sum of the digits equals $2007 \times 9 = 18063$.

A2. E) We don't need to give all the fractions the same denominator; we can choose the denominators in such a way that they are near 100. So $0,16 = \frac{16}{100}$, $\frac{1}{7} = \frac{14}{98}$, $\frac{5}{33} = \frac{15}{99}$. Since, for $k < n$, $\frac{k}{n} < \frac{k+1}{n+1}$, (because of $\frac{k+1}{n+1} - \frac{k}{n} = \frac{n-k}{n(n+1)} > 0$,) we now can see that $\frac{13}{97} < \frac{14}{98} < \frac{15}{99} < \frac{16}{100} < \frac{17}{101}$.

A3. A) 76 There are $\frac{9\times 8\times 7}{3\times 2\times 1} = 84$ ways to pick three points from the nine grid points. But if those three points lie on a single line, then those three points do not form a triangle. This occurs eight times; three times horizontally, three times vertically, and two times diagonally. So there are $84 - 8 = 76$ ways to pick three points from the grid points such that they do form a triangle.

A4. B) 7 First we rewrite the equation as $\frac{ab+1}{b} = 13 \times (\frac{ba+1}{a})$. So $a = 13b$, and we see that the only pairs satisfying the conditions are the seven pairs $(13,1), (26,2), \ldots, (91,7)$.

A5. B) First, we see that path A is longer than path B, since $\sqrt{5} > 2$. So path A cannot be the shortest one. Also, path E is longer than path A, since $\sqrt{5} > 1$; path E cannot be the shortest one either. And path D is longer than path B; $\sqrt{13} > \sqrt{10}$. So path D isn't the shortest path.
Now paths B and C remain. They have common segments of lengths $\sqrt{5}$ and 2, so it remains to compare the lengths of the other segments. Path B has a remaining length of $\sqrt{10}+1$, path C has one of $2\sqrt{5}$. As in the figure, $\sqrt{10} + 1 = ADE < ADF < ABC = 2\sqrt{5}$, from which follows that path B has the shortest length.



A6. C) 4 We see that the sequence of final digits is equal to

$$2, 2, 4, 8, 2, 6, 2, 2, 4, 8, 2, 6, 2, \ldots,$$

and that we have a repeating part of length 6, namely $2, 2, 4, 8, 2, 6$. Since $2007 = 334 \times 6 + 3$, the final digit of the 2007th number must be a 4.

A7. C) 1003 Since $9^n + 9^n + 9^n = 3 \times 9^n = 3^{2n+1}$, $n = 1003$ is a solution of our equation.

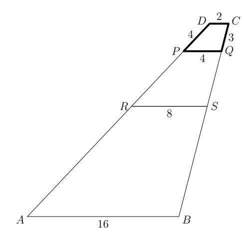
A8. C) 60 Imagine that the six students sit on their chairs in a row. The teachers then can pick two neighbouring students, between which they are going to sit. For the first teacher, there are exactly five ways to pick a chair that is between the chairs of two students. For the second teacher, there are four, and for the third, there are three. So the total number of ways for the teachers to pick their chairs in such a way that every teacher sits between two students, is $5 \times 4 \times 3 = 60$.

B-problems

B1. 415 Considering the worst case scenario, we see that we can get all the cards marked '1' to '9', and 9 of each of the cards marked '10' to '50', without having 10 cards of one kind. In that case, we have $45 + 9 \times 41 = 414$. Now picking one card more (so 415 in total) ensures that we have 10 cards of one kind.

B2. 13 Let P, Q, R, S be the intersections of the two lines where P and R are on DA, and Q and S are on CB, see the figure. Now the quadrilaterals $ABSR, RSQP$ and $PQCD$ are similar. So

$$\frac{|DC|}{|PQ|} = \frac{|PQ|}{|RS|} = \frac{|RS|}{|AB|}.$$



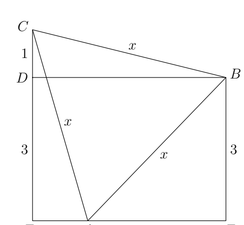
Hence $|PQ|^2 = 2 \times |RS|$, and $|RS|^2 = |PQ| \times 16$, so $|PQ|^3 = 64$, and $|PQ| = 4, |RS| = 8$. So the sides of $RSQP$ are twice as large as the corresponding sides of $PQCD$, and the sides of $ABSR$ are twice as large as the corresponding sides of $RSPQ$. Thus $|RP| = 2 \times |PD|, |AR| = 2 \times |RP| = 4 \times |PD|$, and $|AD| = 7 \times |PD|$, so we conclude that $|PD| = 4$. Similarly, we see that $|QC| = 3$.
Thus the perimeter of $PQCD$ equals 13.

B3. 120 Recursively, we compute

$$\begin{array}{lcl}
8 \otimes 5 = (5+3) \otimes 5 & \dfrac{5 \otimes (5+3)}{5 \otimes 3} = \dfrac{8}{3} & 8 \otimes 5 = \dfrac{8}{3} \times (5 \otimes 3) \\
5 \otimes 3 = (3+2) \otimes 3 & \dfrac{3 \otimes (3+2)}{3 \otimes 2} = \dfrac{5}{2} & 5 \otimes 3 = \dfrac{5}{2} \times (3 \otimes 2) \\
3 \otimes 2 = (2+1) \otimes 2 & \dfrac{2 \otimes (2+1)}{2 \otimes 1} = \dfrac{3}{1} & 3 \otimes 2 = 3 \times (2 \otimes 1) \\
2 \otimes 1 = (1+1) \otimes 1 & \dfrac{1 \otimes (1+1)}{1 \otimes 1} = \dfrac{2}{1} & 2 \otimes 1 = 2 \times (1 \otimes 1)
\end{array}$$

Since $1 \otimes 1 = 1 + 2 = 3$, we have $2 \otimes 1 = 2 \times 3 = 6$, then $3 \otimes 2 = 3 \times 6 = 18$, and $5 \otimes 3 = \frac{5}{2} \times 18 = 45$, and finally, $8 \otimes 5 = \frac{8}{3} \times 45 = 120$.

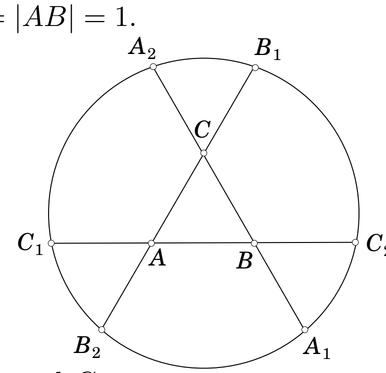
B4. $\frac{2}{3}\sqrt{39}$ Let x denote the side of the equilateral triangle. By repeated application of Pythagoras' Theorem, we get $x^2 = |BD|^2 + 1 = (|EA| + |AF|)^2 + 1 = (\sqrt{x^2 - 4^2} + \sqrt{x^2 - 3^2})^2 + 1 = 2x^2 - 25 + 2\sqrt{x^2 - 4^2}\sqrt{x^2 - 3^2} + 1$. Rewriting this equation gives us $2\sqrt{x^2 - 4^2}\sqrt{x^2 - 3^2} = -x^2 + 24$. Then we square both sides: $4(x^4 - 25x^2 + 144) = x^4 - 48x^2 + 576$. Thus $3x^4 - 52x^2 = 0$. Now the only positive solution is $x = \sqrt{\frac{52}{3}} = \frac{2}{3}\sqrt{39}$.



Dutch Mathematical Olympiad
Round 2
Friday, September 14, 2007
Available time: 3 hours

- Writing down just the answer itself is not sufficient; you also need to describe the way you solved the problem.
- Calculators and formula sheets are not allowed; you are only allowed to use a pen, a compass and a ruler or set square. And of course your common sense.
- Please write the solutions of each problem on a different sheet of paper.
- Good luck!

1. Consider the equilateral triangle ABC with $|BC| = |CA| = |AB| = 1$.
On the extension of side BC, we define points A_1 (on the same side as B) and A_2 (on the same side as C) such that $|A_1B| = |BC| = |CA_2| = 1$. Similarly, we define B_1 and B_2 on the extension of side CA such that $|B_1C| = |CA| = |AB_2| = 1$, and C_1 and C_2 on the extension of side AB such that $|C_1A| = |AB| = |BC_2| = 1$.
Now the circumcentre of $\triangle ABC$ is also the centre of the circle that passes through the points A_1, B_2, C_1, A_2, B_1 and C_2.
Calculate the radius of the circle through A_1, B_2, C_1, A_2, B_1 and C_2.

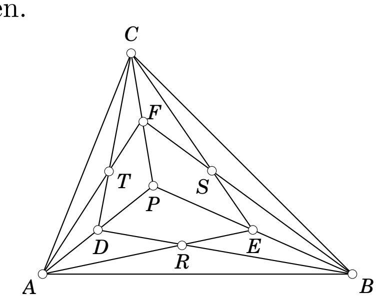


2. Is it possible to partition the set $A = \{1, 2, 3, \ldots, 32, 33\}$ into eleven subsets that contain three integers each, such that for every one of these eleven subsets, one of the integers is equal to the sum of the other two? If so, give such a partition; if not, prove that such a partition cannot exist.

3. Does there exist an integer having the form $444\cdots4443$ (all fours, and ending with a three) that is divisible by 13? If so, give an integer having that form that is divisible by 13; if not, prove that such an integer cannot exist.

4. Determine the number of integers a satisfying $1 \leqslant a \leqslant 100$ such that a^a is a perfect square. (And prove that your answer is correct.)

5. A triangle ABC and a point P inside this triangle are given.
Define D, E and F as the midpoints of AP, BP and CP, respectively. Furthermore, let R be the intersection of AE and BD, S the intersection of BF and CE, and T the intersection of CD and AF.
Prove that the area of hexagon $DRESFT$ is independent of the position of P inside the triangle.

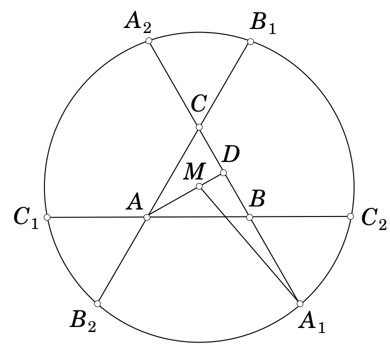


Solutions of Round 2 of the Dutch Mathematical Olympiad 2007.

Problem 1:

Let M be the circumcentre of $\triangle ABC$ and D be the midpoint of BC.

Draw AD through M, and draw MA_1. Since AD is a median, $|MD| = \frac{1}{3}|AD| = \frac{1}{6}\sqrt{3}$ and $|BD| = \frac{1}{2}$. Now, applying Pythagoras' Theorem to $\triangle MDA_1$, we find $|MA_1|^2 = |MD|^2 + |DA_1|^2 = (\frac{1}{6}\sqrt{3})^2 + (\frac{3}{2})^2 = \frac{7}{3}$. So $|MA_1| = \frac{1}{3}\sqrt{21}$. (Similarly, one can compute the distance from M to the other points, yielding $\frac{1}{3}\sqrt{21}$ every time.) So the radius of the circle passing through the points $A_1, B_2, C_1, A_2, B_1, C_2$ is equal to $\frac{1}{3}\sqrt{21}$.



Problem 2:

Suppose that we have such a partition into eleven subsets. Then for each of those eleven subsets $\{a, b, c\}$ we have that, say, $a + b = c$. So $a + b + c = 2c$, and the sum of the integers in each subset is even. Hence also the sum of $1, 2, \ldots, 32, 33$ must be even.
But since $1 + 2 + 3 + \cdots + 33 = \frac{1}{2} \times 33 \times (33 + 1) = 33 \times 17$ is an odd number, we have a contradiction. So there doesn't exist a partition with the desired properties.

Problem 3:

Suppose that such an integer $\underbrace{444\cdots4443}_{k}$ exists and that it is divisible by 13, then substracting 13 from that integer gives another integer that is divisible by 13. So $444\cdots4430$ is divisible by 13. Since $444\cdots4430 = 444\cdots443 \times 10$, and since 10 is not divisible by the prime 13, $\underbrace{444\cdots443}_{k-1}$, the integer with one 4 less, must also be divisible by 13. We can repeat this argument until we get the integer 43, from which we conclude that 43 must also be divisible by 13. But 43 is not a multiple of 13, hence the integer we started with cannot be divisible by 13 either. So there is no integer having that form that is divisible by 13.
(When the amount of 4's equals 0, we have the number 3, which obviously isn't a multiple of 13 either.)

Problem 4:

In the prime factorisation of a perfect square, every prime factor occurs an even number of times. Also, if in the prime factorisation of an integer every prime factor occurs an even number of times, then that integer must be a perfect square.

We consider two cases; namely a being even, or a being odd.

- Suppose a is even, and write $a = 2c$, then $a^a = (2c)^{2c} = (2^c c^c)^2$. So in this case, a^a always is a perfect square.
- Now suppose that a is odd, and write $a = 2c + 1$. Then

  $$a^a = (2c+1)^{2c+1} = (2c+1)^{2c}(2c+1) = \big((2c+1)^c\big)^2(2c+1).$$

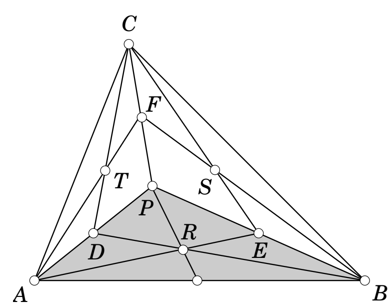
  Now $\big((2c+1)^c\big)^2$ is a perfect square, so a^a is a perfect square if and only if every prime factor of $(2c+1)$ occurs an even number of times, so if and only if $(2c+1)$ itself is a perfect square.

So we conclude that every even integer and every odd perfect square no larger than 100 satisfy the condition, and there are $50 + 5 = 55$ of those.

Problem 5:

We denote the area of $\triangle ABC$ as $[ABC]$. Similarly,

we denote the area of a quadrilateral $ABCD$ as $[ABCD]$.



Let us first consider $\triangle ABP$. The segments AE and BD are two medians of this triangle, so R is the centroid of this triangle. Drawing the third median from P through R, we see that the triangle is divided into 6 triangles with equal area. So $[DREP] = \frac{2}{6}[ABP]$.

In a similar way, we find for $\triangle BCP$ and $\triangle CAP$ respectively that $[ESFP] = \frac{1}{3}[BCP]$ and $[FTDP] = \frac{1}{3}[CAP]$. Hence

$$[DRESFT] = \tfrac{1}{3}([ABP] + [BCP] + [CAP]) = \tfrac{1}{3}[ABC],$$

which is independent of the location of P.

Team Selection Test

Valkenswaard, June 7, 2008

1. Find all funtions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that satisfy
$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$
for all $n \in \mathbb{Z}_{>0}$.

2. Julian and Johan are playing a game with an even number of cards, say $2n$ cards, ($n \in \mathbb{Z}_{>0}$). Every card is marked with a positive integer. The cards are shuffled and are arranged in a row, in such a way that the numbers are visible. The two players take turns picking cards. During a turn, a player can pick either the rightmost or the leftmost card.
Johan is the first player to pick a card (meaning Julian will have to take the last card). Now, a player's *score* is the sum of the numbers on the cards that player acquired during the game.
Prove that Johan can always get a score that is at least as high as Julian's.

3. Let m, n be positive integers. Consider a sequence of positive integers $a_1, a_2, \ldots, a_n$ that satisfies $m = a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. Then define, for $1 \leq i \leq m$,
$$b_i = \#\{j \in \{1, 2, \ldots, n\} : a_j \geq i\},$$
so b_i is the number of terms a_j of the given sequence for which $a_j \geq i$. Similarly, we define, for $1 \leq j \leq n$,
$$c_j = \#\{i \in \{1, 2, \ldots, m\} : b_i \geq j\},$$
thus c_j is the number of terms b_i in the given sequence for which $b_i \geq j$.
E.g.: If a is the sequence $5, 3, 3, 2, 1, 1$ *then b is the sequence* $6, 4, 3, 1, 1$.
   (a) Prove that $a_j = c_j$ for $1 \leq j \leq n$.
   (b) Prove that for $1 \leq k \leq m$: $\sum_{i=1}^{k} b_i = k \cdot b_k + \sum_{j=b_k+1}^{n} a_j$.

4. Let n be positive integer such that $\sqrt{1 + 12n^2}$ is an integer.
Prove that $2 + 2\sqrt{1 + 12n^2}$ is the square of an integer.

5. Let $\triangle ABC$ be a right triangle with $\angle B = 90°$ and $|AB| > |BC|$; and let Γ be the semicircle with diameter AB that lies on the same side as C. Let P be a point on Γ such that $|BP| = |BC|$ and let Q be on AB such that $|AP| = |AQ|$.
Prove that the midpoint of CQ lies on Γ.

Solutions of the Team Selection Test 2008

Problem 1. If $f(m) = f(n)$, then $3m = 3n$, thus $m = n$. So f is injective. Now we use induction on n to show that $f(n) = n$ for all n.
Since $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, $f(1) \geq 1$, $f(f(1)) \geq 1$ and $f(f(f(1))) \geq 1$. Substituting $n = 1$ in the functional equation yields

$$f(f(f(1))) + f(f(1)) + f(1) = 3,$$

so equality must hold everywhere, meaning that $f(1) = f(f(1)) = f(f(f(1))) = 1$.
Let $k \geq 2$, and for any $n < k$, suppose that $f(n) = n$. Then injectivity implies for all $m \geq k$ that $f(m) \geq k$. So in particular, $f(k) \geq k$, which implies $f(f(k)) \geq k$, which again implies $f(f(f(k))) \geq k$. Substituting $n = k$ in the functional equation now yields

$$f(f(f(k))) + f(f(k)) + f(k) = 3k.$$

Thus equality must hold everywhere; $f(k) = f(f(k)) = f(f(f(k))) = k$. This completes our induction.

The above proves that the only function that can satisfy the functional equation is $f(n) = n$. Checking this,

$$f(f(f(n))) + f(f(n)) + f(n) = n + n + n = 3n,$$

also proves that f also satisfies the functional equation. □

Problem 2. Let $2n$ be the number of cards, and assume that the cards are marked with (from left to right) $a_1, a_2, \ldots, a_{2n}$. We'll prove by induction on n that Johan can always pick either the odd cards $a_1, a_3, \ldots, a_{2n-1}$, or the even cards $a_2, a_4, \ldots, a_{2n}$. This is clear for $n = 1$; Johan picks the first card if he wants the odd cards, and he picks the second if he wants the even cards.
So now suppose that the above statement is true for $n = k$, and consider the situation with $2k + 2$ cards, $a_1, a_2, \ldots, a_{2k+2}$. If Johan wants the odd cards, he first picks a_1, and Julian will have to either pick a_2 or a_{2k+2} in that case. In the first case, the sequence of cards after 2 turns becomes $\{b_i\}$ where $b_i = a_{i+2}$, and according to the induction hypthesis, Johan can get the odd b_i, which in combination with a_1, will get him all the odd a_i. In the second case, the sequence of cards becomes $\{b_i\}$ where $b_i = a_{i+1}$, and again, according to the induction hypothesis, Johan can get the even b_i, which in combination with a_1, will get him all odd a_i. So Johan can pick the odd cards. In a similar way we can prove that Johan can pick the even cards. This completes the induction.

Now if the sum of the numbers on the odd cards is larger than or equal to the sum of the numbers on the even numbers, Johan picks the odd cards. Otherwise he picks the even cards. This ensures that Johan will get a score at least as high as Julian's. □

Problem 3.

(a) *Solution 1.* Note that, for $1 \leq i \leq m, 1 \leq j \leq n$:

$$\begin{aligned} a_j \geq i & \iff \\ a_1, a_2, \ldots, a_j \geq i & \iff \\ \text{at least } j \text{ terms of } a \text{ are greater than or equal to } i & \iff \\ b_i \geq j & \iff \\ b_1, b_2, \ldots, b_i \geq j & \iff \\ \text{at least } i \text{ terms of } b \text{ are greater than or equal to } j & \iff \\ c_j \geq i. & \end{aligned}$$

Now let $1 \leq j \leq n$, and $i = a_j$. Then $a_j \geq i$, from which follows that $c_j \geq i = a_j$. Similarly, if $i = c_j$, then similarly, we get $a_j \geq i = c_j$. Thus $a_j = c_j$.

Solution 2. Since the sequences are non-increasing, $b_i = \max\{l : a_l \geq i\}$ and $c_j = \max\{i : b_i \geq j\}$. Thus, for $1 \leq j \leq n$:

$$c_j = \max\{i : b_i \geq j\} = \max\{i : \max\{l : a_l \geq i\} \geq j\}.$$

For fixed i, we have

$$\max\{l : a_l \geq i\} \geq j \quad \iff \quad a_j \geq i,$$

so $c_j = \max\{i : a_j \geq i\} = a_j$.

(b) *Solution 1.* For $1 \leq k \leq m$,

$$\sum_{i=1}^{k}(b_i - b_k) = \sum_{i=1}^{k}\left(\#\{l : a_l \geq i\} - \#\{l : a_l \geq k\}\right) = \sum_{i=1}^{k}\#\{l : k > a_l \geq i\}.$$

Any term a_l with $k > a_l$ is counted exactly a_l times in this sum (for every $i \leq a_l$), so this sum must equal the sum of all such a_l. By solution 1 of a), we now have $k \leq a_l$ if and only if $l \leq b_k$, so $k > a_l$ if and only if $l \leq b_k$. Thus,

$$\sum_{i=1}^{k}(b_i - b_k) = \sum_{l:k>a_l} a_l = \sum_{l=b_k+1}^{n} a_l,$$

which is what we needed to prove.

Solution 2. By induction on k. If $k=1$, then we need to prove

$$b_1 = b_1 + \sum_{j=b_1+1}^{n} a_j,$$

which is obviously true since $b_1 = \#\{j : a_j \geq 1\} = n$, so the sum on the right hand side is empty. So suppose that what we need to prove is true for $k = s, 1 \leq s \leq m-1$. Then

$$\begin{aligned}
\sum_{i=1}^{s+1} b_i &= \sum_{i=1}^{s} b_i + b_{s+1} \\
&\overset{\text{IH}}{=} s \cdot b_s + \sum_{j=b_s+1}^{n} a_j + b_{s+1} \\
&= (s+1) \cdot b_{s+1} + \sum_{j=b_{s+1}+1}^{n} a_j + s(b_s - b_{s+1}) - \sum_{j=b_{s+1}+1}^{b_s} a_j \\
&= (s+1) \cdot b_{s+1} + \sum_{j=b_{s+1}+1}^{n} a_j + s(b_s - b_{s+1}) - \sum_{j=b_{s+1}+1}^{b_s} s \\
&= (s+1) \cdot b_{s+1} + \sum_{j=b_{s+1}+1}^{n} a_j.
\end{aligned}$$

□

Problem 4. Let a be such that $1 + 12n^2 = a^2$. Rewrite this as

$$12n^2 = a^2 - 1 = (a+1)(a-1).$$

Note that the left hand side is even, hence so is the right hand side, and a has to be odd. Since the left hand side has an even number of factors 2, both $a+1, a-1$ have an odd number of factors 2, since one of these two has exactly one factor 2. Also, $(a+1, a-1) = 2$, from which follows that for any odd prime divisor p of the left hand side, p divides exactly one of $a+1, a-1$. Thus, if $p \geq 5$, the right hand side has an even number of factors p, and if $p = 3$, the right hand side has an odd number of factors p. Hence we have the following two possibilities:

$$a + 1 = 6b^2 \quad \text{and} \quad a - 1 = 2c^2$$

for certain integers b, c such that $bc = n$, or

$$a + 1 = 2b^2 \quad \text{and} \quad a - 1 = 6c^2$$

for certain integers b, c such that $bc = n$.

Consider the first case. Then $3 | a+1$, so $a - 1 \equiv 1 \mod 3$. Thus $c^2 \equiv 2 \mod 3$, which is a contradiction. Hence the second case must hold. In that case,

$$2 + 2\sqrt{1 + 12n^2} = 2 + 2a = 2(a+1) = 4b^2 = (2b)^2,$$

which was what we needed to prove. □

Problem 5. *Solution 1.* Let S be the intersection of Γ en CQ. We need to prove that $|QS| = |SC|$.

Note that BC is tangent to Γ, so $\angle CBP = \angle BAP = \angle QAP$. Since both $\triangle CBP$ and $\triangle QAP$ are both isosceles, they're similar. Let $\alpha = \angle BCP$, then

$$\alpha = \angle BCP = \angle CPB = \angle QPA = \angle AQP.$$

Now note that

$$\angle CPQ = \angle CPB + \angle BPQ = \angle QPA + \angle BPQ = \angle BPA = 90^\circ.$$

So P, just like B, lies on the circle with diameter CQ. S lies on this diameter, and all that's left to show, is

$$\angle BSP = 2\alpha = 2\angle BCP.$$

Considering the cyclic quadrilateral $QBCP$, we see that $\angle CPB = \angle CQB$, so

$$2\alpha = \angle CQB + \angle AQP = 180^\circ - \angle PQC = 180^\circ - \angle PBC = 90^\circ + \angle QBC - \angle PBC = 90^\circ + \angle QBP.$$

Considering the cyclic quadrilateral $ABSP$, with diameter AB, we see that

$$90^\circ + \angle QBP = 90^\circ + \angle ABP = \angle BSA + \angle ASP = \angle BSP,$$

from which follows that

$$2\alpha = \angle BSP,$$

which is what we needed to prove. □

Solution 2. First of all, as in solution 1, the two isosceles triangles are similar. Since BC is orthogonal to AB, one of the triangles is rotated by 90° with respect to the other. Now let l_1 be the angle bisector of $\angle PBC$ and l_2 be the angle bisector $\angle PAQ$, and note that these angle bisectors are perpendicular. Let T be the intersection of l_1 and l_2. Then $\angle ATB = 90^\circ$, so T lies on Γ.

Note that P is the image of C under reflection in l_1, and Q is the image of P under reflection in l_2. The composition of these two reflections is the rotation about T by $2 \cdot 90 = 180$ degrees. This rotation sends C to Q, so CTQ is a straight line. Thus $T = S$. Also, $|TC| = |TQ|$, so S is the midpoint of CQ. □

Dutch Mathematical Olympiad
Round 1
Friday, January 25, 2008
Available time: 2 hours

Please read the following before you start:

- The A-problems are multiple choice questions. Only one of the five given options is correct. Please state clearly behind which letter the correct solution is stated. You get 2 points for each correct answer.
- The B-problems are open questions, which have a number as answer. You get 5 points for each correct answer. Please work accurately, since an error in your calculations can cause your solution to be considered wrong. Please give exact answers, for example $\frac{11}{81}$ or $2+\sqrt{3}$ or $\pi+1$.
- This is a competition, not an exam. So it is to be expected that only few people will get all the answers right, and you don't need to worry if you have only solved a part of the problems.
- The use of formula sheets, calculators and similar tools is **not** allowed.
- The point is that you have fun solving unusual mathematical problems. *Good luck!*

A-problems

A1. Alex, Birgit, Cedric, Dion and Ersin all write their names on a sheet of paper, and they put those five sheets into a large box. They each take one sheet out of the box at random. Now it turns out that Birgit has Alex' sheet, Cedric has Dion's and Dion has Ersin's. Also, Ersin doesn't have Cedric's sheet. Whose sheet does Alex have?

A) Alex' B) Birgit's C) Cedric's D) Dion's E) Ersin's

A2. In a magic 3×3 square, the three row sums, the three column sums and the two diagonal sums are all equal to each other. (A *row sum* being the sum of the numbers on a certain row, etc.) In the magic 3×3 square shown here three numbers have already been filled in. What number must be filled in instead of the question mark?

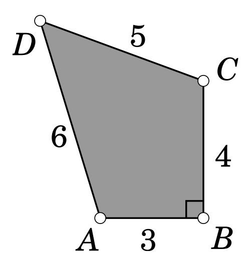
		7
?		
	10	3

A) 2 B) 4 C) 6 D) 8 E) 9

A3. Calculating $6\times 5\times 4\times 3\times 2\times 1$ yields 720. How many divisors does 720 have? (A *divisor* of an integer n is a positive integer by which n is divisible. For example: the divisors of 6 are 1, 2, 3 and 6; the divisors of 11 are 1 and 11.)

A) 6 B) 8 C) 20 D) 30 E) 36

A4. Of a quadrilateral $ABCD$, we know that $|AB|=3$, $|BC|=4$, $|CD|=5$, $|DA|=6$ en $\angle ABC=90°$. ($|AB|$ stands for the length of segment AB, etc.) What is the area of quadrilateral $ABCD$?

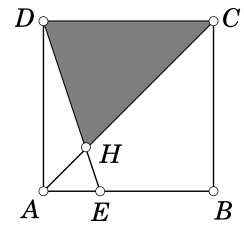


A) 16 B) 18 C) $18\frac{1}{2}$ D) 20 E) $6+5\sqrt{11}$

A5. How many five-digit numbers (like 12345 or 78000; the first digit must be non-zero) are there that end on a 4 and that are divisible by 6?

A) 1500 B) 2000 C) 3000 D) 7500 E) 8998

A6. We have a square $ABCD$ with $|AB| = 3$. On AB, there is a point E such that $|AE| = 1$ and $|EB| = 2$. AC and DE intersect in H. What is the area of triangle CDH?



A) $\frac{9}{8}$ B) 2 C) $\frac{21}{8}$ D) 3 E) $\frac{27}{8}$

A7. The seven blocks $\boxed{S}\boxed{E}\boxed{T}\boxed{T}\boxed{E}\boxed{E}\boxed{S}$ are shuffled. For example, you can get $\boxed{E}\boxed{E}\boxed{E}\boxed{S}\boxed{S}\boxed{T}\boxed{T}$ or $\boxed{T}\boxed{E}\boxed{S}\boxed{E}\boxed{T}\boxed{E}\boxed{S}$.
How many different "words" of length 7 can we get this way? (Any combination of the 7 letters counts as word.)

A) 210 B) 420 C) 840 D) 1260 E) 5040

A8. How many distinct real solutions does the equation $\left((x^2-2)^2-5\right)^2 = 1$ have?

A) 4 B) 5 C) 6 D) 7 E) 8

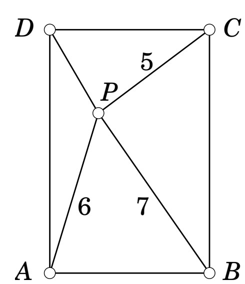
B-problems

B1. We number both the rows and the columns of an 8×8 chessboard with the numbers 1 to 8. A number of grains is placed onto each square, in such a way that the number of grains on a certain square equals the product of its row and column numbers. How many grains are there on the entire chessboard?

B2. We take 50 distinct integers from the set $\{1, 2, 3, \ldots, 100\}$, such that their sum equals 2900. What is the minimal number of *even* integers amongst these 50 numbers?

B3. For a certain x, we have $x + \frac{1}{x} = 5$. Define $n = x^3 + \frac{1}{x^3}$. It turns out that n is an integer.
Calculate n. (Give your answer using decimal notation.)

B4. Inside a rectangle $ABCD$, there is a point P with $|AP| = 6$, $|BP| = 7$ and $|CP| = 5$. What is the length of segment DP?



SOLUTIONS
of the problems of Round 1 of the
Dutch Mathematical Olympiad 2008

A-problems

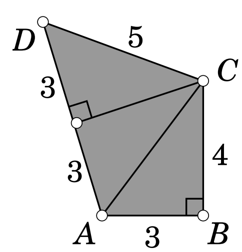
A1. C) Cedric When we put the data into a table, we see that Birgit's and Cedric's sheets haven't been picked yet. Since Ersin didn't pick Cedric's sheet, he must have picked Birgit's. So Alex must have picked Cedric's sheet.

A	B	C	D	E
?	A	D	E	?

A2. B) 4 See the figure. From $F + 10 + 3 = F + D + 7$, we get $D = 6$. Then from $7 + E + 3 = C + D + E = C + 6 + E$, we can deduce that $C = 7 + 3 - 6 = 4$.

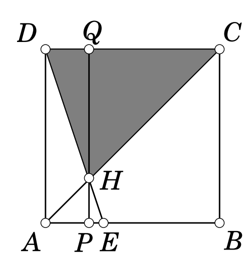
A	B	7
C	D	E
F	10	3

A3. D) 30 The number 720 only has the prime factors 2, 3 and 5. The prime factor 2 occurs four times (once in 2, twice in 4 and once in 6), the prime factor 3 twice (once in 3 and 6), and 5 just once. The divisors without any factors 3 or 5 are 1, 2, 4, 8 and 16. The divisors having one factor 3 and no factors 5 are 3, 6, 12, 24, 48. And the divisors having two factors 3 and no factors 5 are 9, 18, 36, 72 and 144. So 720 has 15 divisors that do not have factors 5. Multiplying all of these divisors by 5 gives us the other 15 divisors, which makes 30 in total.
Alternative solution: Every divisor of $720 = 2^4 \times 3^2 \times 5^1$ can be written as $2^a \times 3^b \times 5^c$ with 5 possibilities for a (being 0 to 4), 3 possibilities for b (being 0 to 2) and 2 for c (being 0 and 1). So we conclude that 720 has $5 \times 3 \times 2 = 30$ divisors.



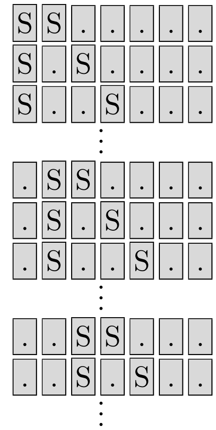
A4. B) 18 According to Pythagoras' Theorem, we have $|AC| = 5$. So triangle ACD is isosceles with base AD. In this triangle, the altitude from C divides the triangle into two triangles with sides 3, 4 and 5, and we can divide quadrilateral $ABCD$ in three triangles with sides 3, 4 and 5. So its area must be equal to $3 \times 6 = 18$.

A5. C) 3000 If x is a positive multiple of 6 that ends with a 4, then the next multiples of 6 end with a 0 $(x + 6)$, a 6 $(x + 12)$, a 2 $(x + 18)$, an 8 $(x + 24)$, a 4 $(x + 30)$, so the next multiple of 6 that ends with a 4 is $x + 30$. So any 30 consecutive positive integers must contain exactly one integer with the desired properties. How many such integers lie between 10000 and 99999? Since we have 90000 consecutive positive integers, we find $90000 \div 30 = 3000$ such integers amongst them.



A6. E) $\frac{27}{8}$ Draw a line through H parallel to AD, and let P and Q be the intersections of this line with AB and CD, respectively. Now we have $|HP| : |HQ| = |AE| : |CD| = 1 : 3$, so $|HQ| = \frac{3}{4} \times |PQ| = \frac{3}{4} \times 3 = \frac{9}{4}$. So the area of triangle CDH equals $\frac{1}{2} \times 3 \times \frac{9}{4} = \frac{27}{8}$.

A7. A) 210 We have $6+5+4+3+2+1=21$ (or $\binom{7}{2}$) possibilities to arrange the S-blocks onto the 7 places (see figure). For each choice, we have $4+3+2+1=10$ (or $\binom{5}{2}$) possibilities to arrange the T-blocks on the remaining 5 places; after which the positions of the E-blocks are determined. So we have $21\times 10=210$ possibilities.



Alternative solution: If all the blocks were different, we would have got 7! possibilities. But the three E-blocks aren't different, so we end up counting each word 3! times this way. Similarly for the two S-blocks and the two T-blocks. So we find $\frac{7!}{3!\times 2!\times 2!}=\frac{7\times 6\times 5\times 4\times \cancel{3!}}{\cancel{3!}\times 4}=7\times 6\times 5=210$ different words.

A8. B) 5 This equation is equivalent to

$$(x^2-2)^2-5=1 \quad \text{or} \quad (x^2-2)^2-5=-1.$$

The first is equivalent to $x^2-2=\sqrt{6}$ of $x^2-2=-\sqrt{6}$, with 2 and 0 solutions respectively (since $-\sqrt{6}+2<0$).
The latter is equivalent to $x^2-2=2$ of $x^2-2=-2$, with 2 and 1 solution(s) respectively. So we have $2+0+2+1=5$ solutions in total.

B-problems

B1. 1296 In the first column, we have, successively, $1\times 1, 1\times 2, 1\times 3, \ldots, 1\times 8$ grains.
So, in the first column: $1\times(1+2+3+4+5+6+7+8)$.
In the second column: $2\times(1+2+3+4+5+6+7+8)$.
In the third column is: $3\times(1+2+3+4+5+6+7+8)$.
$\vdots$
Finally, in the eighth column: $8\times(1+2+3+4+5+6+7+8)$.
So, in total: $(1+2+3+4+5+6+7+8)\times(1+2+3+4+5+6+7+8)$.
Since $1+2+3+4+5+6+7+8=\frac{1}{2}\times 8\times(1+8)=36$, there are $36^2=1296$ grains on the board.

B2. 6 The 50 odd integers from the set $\{1,2,3,\ldots,100\}$ sum up to $\frac{1}{2}\times 50\times(1+99)=2500$, which is still 400 short of 2900. Now exchange the smallest odd integers for the largest even integers, in pairs, since 400 is even. First exchanging 1 and 3 for 100 and 98 makes the sum equal to 2694. The next step gives us $2694-5-7+96+94=2872$. Which is still less than 2900, so we require another exchange. Now exchanging 9 and 11 for 20 and 28 works, making the sum 2900 with 6 even integers, showing along the way that that is the minimal number of even integers we need to do so.

B3. 110 We know that x must satisfy $x+\frac{1}{x}=5$, so $x^2-5x+1=0$, from which follows that $x=x_{1,2}=\frac{5\pm\sqrt{21}}{2}$. Note that $x_1x_2=1$. Now we have $x^3+x^{-3}=x_1^3+x_2^3=\frac{1}{8}\left(\left(5+\sqrt{21}\right)^3+\left(5-\sqrt{21}\right)^3\right)=\frac{1}{8}\left(\left(5^3+3\cdot 5^2\cdot\sqrt{21}+3\cdot 5\cdot\sqrt{21}^2+\sqrt{21}^3\right)+\left(5^3-3\cdot 5^2\cdot\sqrt{21}+3\cdot 5\cdot\sqrt{21}^2-\sqrt{21}^3\right)\right)=\frac{2}{8}\left(5^3+3\cdot 5\cdot 21\right)=110.$
Alternative solution: From $(x+\frac{1}{x})^3=x^3+3x^2(\frac{1}{x})+3x(\frac{1}{x})^2+(\frac{1}{x})^3=x^3+3x+\frac{3}{x}+\frac{1}{x^3}$ we can deduce that $x^3+\frac{1}{x^3}=(x+\frac{1}{x})^3-3(x+\frac{1}{x})=5^3-3\times 5=110.$

B4. $2\sqrt{3}$ Let Q, R, S, T be the orthogonal projections of P on AB, BC, CD, DA, respectively. Then we have
$|AQ|^2+|QP|^2=36$ and $|BQ|^2+|SP|^2=25$ (since $|BQ|=|CS|$),
so $|AQ|^2+|QP|^2+|BQ|^2+|SP|^2=61$. We also have $|BQ|^2+|QP|^2=49$.
So $|DS|^2+|SP|^2=|AQ|^2+|SP|^2=61-49=12$ and $|DP|=\sqrt{12}=2\sqrt{3}$.